Practice Midterm 1 Solutions, 18.100C, Fall 2012

1

Suppose F is a field, and $x, y \in F$ satisfy $x^2 = y^2$. We need to show that either $x = y$ or $x = -y$. For the sake of sparing the proliferation of parenthesis, we will adopt the standard conventions about multiplication and addition, i.e. $x + y \cdot z$ is $x + (y \cdot z)$ rather than $(x + y) \cdot z$.

Now consider the quantity $(x + (-y)) \cdot (x + y)$. Using the distributive property, this is equal to $(x + (-y))x + (x + (-y))y$. Using distributivity again, we get $(x^2 + (-y) \cdot x) + (x \cdot y + (-y) \cdot y)$. Using associativity of addition and commutativity of multiplication, we can rearrange this to $x^2 + ((x \cdot y + x \cdot (-y)) + (-y) \cdot y)$. Now, we have

$$x \cdot y + x \cdot (-y) = x \cdot (y + (-y)) = x \cdot 0 = 0$$

Where the first step follows from distributivity, the second from the definition of $-y$, and the third from noting that that $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ and adding $-(x \cdot 0)$ to both sides. Similarly, $y \cdot (-y) + y^2 = 0$, and adding $-y^2$ on the right to both sides gives $(-y) \cdot y = -y^2$. Going back, we have

$$\begin{aligned}(x - y) \cdot (x + y) &= x^2 + ((x \cdot y + x \cdot (-y)) + (-y) \cdot y) \\ &= x^2 + ((0) + (-y^2)) = x^2 + (-y^2)\end{aligned}$$

But by assumption $x^2 = y^2$, so this is equal to $y^2 + (-y^2) = 0$. In other words, $(x - y) \cdot (x + y) = 0$.

If $x = y$, then we are done. So suppose $x \neq y$. Then $x - y \neq 0$, since if $x - y = 0$ we have $y = 0 + y = (x - y) + y = x + (y - y) = x + 0 = x$

using, respectively, definition of 0, substitution, associativity of addition, definition of $-y$, and definition of 0. So there must exist a multiplicative inverse $(x-y)^{-1}$. Then we have

$$0 = (x-y)^{-1} \cdot 0 = (x-y)^{-1}((x-y)(x+y) =$$
$$((x-y)^{-1}(x-y))(x+y) = 1 \cdot (x+y) = x+y$$

The first equality is a fact we already proved, the second is just substitution, the third is associativity of multiplication, the fourth is the definition of multiplicative inverse, and the fifth is the definition of 1.

Now we get

$$x = x + 0 = x + (y - y) = (x + y) - y = 0 - y = -y$$

Using the definition of 0, definition of $-y$, associativity of addition, substitution, and definition of 0. In other words, if $x \neq y$, then $x = -y$, which is what we wanted to prove.

2

Suppose E is a finite dense subset of X. Then E contains no limit points. To see this, suppose $x \in X$ is a limit point of E. Then by Theorem 2.20 of Rudin, every neighbourhood of x has to contain infinitely many points of E. But E only has finitely many points, so this is obviously impossible. Since E has no limit points, it is vacuously true that it contains all its limit points, so E is closed and $E = \overline{E}$. But E is dense in X, so $\overline{E} = X$. This means that $X = E$, so X itself must be finite since E is.

3

We will show that $\mathbb{Z}$ is not compact in the p-adic topology by constructing an infinite set with no limit points. We begin with an elementary

Lemma 1: Let $m \in \mathbb{N}$ and k the largest natural number with $p^k | m$. Let l be natural number with $m < p^l$. Then p^k is the largest power of p dividing $m + p^l$

Proof: We have $p^k \leq m < p^l$, so $k < l$ and $p^k | p^l$, hence p^k divides $m + p^l$. Since $k < l$, $k+1 \leq l$, so $p^{k+1} | p^l$. But then p^{k+1} cannot divide $m + p^l$, since then it would divide $(m + p^l) - p^l = m$, a contradiction.

Now for $n \in \mathbb{N}$, we introduce the finite sum $s_n = \sum_{i=0}^{i=n} p^{2i}$, where p is the prime with respect to which we define the p-adic metric on $\mathbb{Z}$. Let $S = \{s_n | n \in \mathbb{N}\}$ be the set of all these numbers. S is obviously infinite, so if we can show that S has no limit points we are done. To do so, we need another elementary

Lemma 2: $2s_n < p^{2n+2}$

Proof: Recall that by the formula for the sum of a geometric series, $s_n = (p^{2n+2} - 1)/(p^2 - 1)$. Since $p \geq 2$ we have $p^2 > 3$, and so $p^{2n+4} - 3p^{2n+2} + 2 = p^{2n+2}(p^2 - 3) + 2 > 0$. Using these facts, the lemma follows from simple algebra:

$$0 < p^{2n+4} - 3p^{2n+2} + 2 \implies 2p^{2n+2} - 2 < p^{2n+2}(p^2 - 1) \implies 2 \cdot \frac{p^{2n+2} - 1}{p^2 - 1} < p^{2n+2}$$

$$\implies 2s_n < p^{2n+2}$$

With these two lemmas, we can show that S has no limit points. Let $x \in \mathbb{Z}$. Pick some n sufficiently large that $|x| < s_n$, which is always possible since $s_n > n$. Let $k \in \mathbb{N}$ be the largest power of p dividing $s_n - x$, i.e. $d(x, s_n) = p^{-k}$. $s_n > |x|$ means that $s_n - x \in \mathbb{N}$ is positive, and $s_n - x < 2s_n < p^{2n+2}$, where the last step follows from Lemma 2. By Lemma 1, the largest power of p dividing $(s_n - x) + p^{2n+2}$ is k. But $(s_n - x) + p^{2n+2} = (s_n + p^{2n+2}) - x = s_{n+1} - x$, and so $d(x, s_{n+1}) = p^{-k}$.

Applying this argument again, $d(x, s_{n+2}) = p^{-k}$, and indeed by induction $d(x, s_m) = p^k$ for all $m \geq n$. Now let $\epsilon > 0$ be a real number with $\epsilon < p^{-k}$. Then if $s_m \in S \cap N_\epsilon(x)$, we must have $m < n$. So this neighbourhood contains only finitely many points of S, and hence x is not a limit point of S. Since x was arbitrary, this means that S has no limit points and $\mathbb{Z}$ is not compact.

4

We have $K \subset X$ compact, $E \subset X$ closed, and $K \cap E = \emptyset$. We need to show that there exists $D > 0$ such that $d(x, y) \geq D$ for $x \in X$, $y \in Y$.

Suppose not; we will derive a contradiction. Then for every $D > 0$, we can find $x \in K, y \in E$ such that $d(x, y) < D$. For each $n \in \mathbb{N}$, pick some $x_n \in K$, and $y_n \in E$, such that $d(x_n, y_n) < 1/n$. Let $S = \{x_n | n \in \mathbb{N}\}$ be the set of all the x_n's. We consider two cases.

Case 1: S is finite. This means there is a finite collection of points $\{z_1, z_2 \ldots z_M\}$ such that for all $i \in \mathbb{N}$, there exists $j \leq M$ such that $x_i = z_j$. Now define the set $N_j \subset \mathbb{N}$, $1 \leq j \leq M$, by $N_j = \{i \in \mathbb{N} | x_i = z_j\}$. By assumption, $\mathbb{N}$ is the union of the N_j's. But since there are only finite many N_j's, at least one of them must be infinite, since otherwise $\mathbb{N}$ would be a finite union of finite sets, and hence finite. So pick a j such that N_j is infinite.

I claim that z_j is a limit point of E. Fix any $\epsilon > 0$; we will find $y \in N_\epsilon(z_j) \cap E$. Take some $N_0 \in \mathbb{N}$ with $1/N_0 < \epsilon$. Since N_j is infinite and there are obviously only finite many positive integers less than N_0, there must exist some $n \in N_j$ with $n > N_0$. Then $1/n < 1/N_0$, and by definition $z_j = x_n$, and

$$d(z_j, y_n) = d(x_n, y_n) < 1/n < \epsilon$$

This means that z_j is a limit point of E. Since E is closed, $z_j \in E$. On the other hand, $z_j = x_n$ and $x_n \in K$ by construction. Thus $z_j \in K \cap E$, but $K \cap E = \emptyset$, contradiction.

Case 2: S is infinite. S is an infinite subset of a compact set K, hence must have a limit point x. x is then obviously a limit point of K, and since compact sets are closed we must have $x \in K$. We will show that it is also a limit point of E. Let $\epsilon > 0$; as before, we want to find $y \in E$ with $d(x, y) < \epsilon$.

Since x is a limit point of E, the neighbourhood $N_{\epsilon/2}(x)$ must contain infinitely many points of S. Pick $N \in \mathbb{N}$ with $1/N < \epsilon/2$. Since there are only finitely many x_n with $n \leq N$, and $S \cap N_{\epsilon/2}(x)$ is infinite, there must be an $x_n \in S \cap N_{\epsilon/2}(x)$ with $n > N$. Then using the triangle inequality we have

$$d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) < \epsilon/2 + 1/n < \epsilon/2 + \epsilon/2 = \epsilon$$

Since ϵ was arbitrary, this means that x is a limit point of E. But E is closed, so $x \in E$, and we already knew that $x \in K$, contradiction.