Problem Set 9 Solutions, 18.100C, Fall 2012

November 30, 2012

1

Write $\mathcal{B}^1 := \mathcal{B}^1([a,b])$, $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$ the uniform metric on real bounded functions, and $d^1(f,g) = d(f,g) + d(f',g')$ the given metric on $\mathcal{B}^1$. Note that if $f, g \in \mathcal{B}^1$, then f, f', g, g' are all bounded, and $d(f,g) \leq d^1(f,g)$ and $d(f',g') \leq d^1(f,g)$.

Now, let (f_i) be a Cauchy sequence in $\mathcal{B}^1$. Then I claim that (f_i') is Cauchy sequence with respect to d, the uniform metric. Indeed, for any $\epsilon > 0$ pick N sufficiently large that $n, m > N \implies d^1(f_n, f_m) < \epsilon$. But then $d(f_n', f_m') \leq d^1(f_n, f_m) < \epsilon$, so (f_i') is Cauchy.

Thus by Rudin Theorem 7.8, (f_i') is uniformally convergent. By a similar argument, (f_i) is also uniformally convergent, and in particular the sequence $(f_i(x_0))$ converges for any $x_0 \in [a,b]$. Thus by Rudin Theorem 7.17, there exists a differentiable function $f : [a,b] \to \mathbb{R}$ such that $f_i \to f$ uniformally and $f_i' \to f'$ uniformally.

I claim that $f \in \mathcal{B}^1$. To prove this, we need to show that f' is bounded. Pick N sufficiently large that $d(f_N', f') < 1$. Since f_N' is bounded, there exists $M > 0$ such that $|f_N'(x)| < M$ for all $x \in [a,b]$. But then

$$|f'(x)| \leq |f'(x) - f_n'(x)| + |f_N'(x)| < 1 + M$$

So f' is bounded and $f \in \mathcal{B}^1$

Finally, we need to show that $f_i \to f$ in $\mathcal{B}^1$. Let $\epsilon > 0$. Pick N sufficiently large that $n > N$ implies $d(f_n, f) < \epsilon/2$ and $d(f_n', f') < \epsilon$, which we can do by uniform convergence. Then $d^1(f_n, f) = d(f_n, f) + d(f'_n, f') < \epsilon$, so $f_n \to f$ in $\mathcal{B}^1$.

2

We will prove the following more general lemma, which will imply the result

Lemma: Suppose (f_n) is a sequence of functions from $[a, b] \to \mathbb{R}$, and $f_n \to f$ uniformally. Suppose furthermore that for every $p \in (a, b]$ the left limit $f_n(p-)$ exists for all n. Then the left limit $f(-p)$ exists for all $p \in (a, b]$.

A similar result holds for right limits $f(p+)$ for $p \in [a, b)$ with the same proof (after the obvious changes are made).

Before we prove the Lemma, let us show how it implies our result. Let $f : [a, b] \to \mathbb{R}$ be a step function, and $p \in [a, b]$. If p is not one of the finitely many "step" points of f, then f is continuous at p, and so clearly the left and right limits $f(p-)$ and $f(p+)$ exist (and are in fact equal). If p happens to be one of the step points of f, there is some small $\epsilon > 0$ and $c, d \in \mathbb{R}$ such that for $x \in (p - \epsilon, p)$, $f(x) = c$, while for $x \in (p, p + \epsilon)$, $f(x) = d$. But then $f(p-) = c$ and $f(p+) = d$, so the left and right limits both exist.

Hence if (f_n) is a uniformally convergent sequence of step functions, and $p \in [a, b]$ the limits $f_n(p-)$ and $f_n(p+)$ exist, hence by the lemma so do the limits $f(p-)$ and $f(p+)$.

Proof of Lemma: This argument is almost word for word the same as the proof of Rudin Theorem 7.11. Let $p \in (a, b]$, and set $A_n = f_n(p-)$. We first want to show that (A_n) is Cauchy. Let $\epsilon > 0$. Since (f_n) is uniformally convergent it is Cauchy, we there exists $N \in \mathbb{N}$ such that for $n, m > N$ and $x \in [a, b]$ we have

$$|f_n(x) - f_m(x)| \leq \epsilon$$

Then letting $x \to p-$ (i.e. x approaches p from below) we have

$$|A_n - A_m| \leq \epsilon$$

Thus (A_n) is Cauchy, and hence converges to some number A

We need to show that $A = f(p-)$, and in particular that $f(p-)$ exists. Again let $\epsilon > 0$. For any $x \in [a, b]$ and $n \in \mathbb{N}$ we have

$$|f(x) - A| \leq |f(x) - f_n(x)| + |f_n(x) - A_n| + |A_n - A|$$

Since $f_n \to f$ uniformally and $A_n \to A$, we can pick n sufficiently large that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

and

$$|A - A_n| < \frac{\epsilon}{3}$$

Since $f_n(p-) = A_n$, there exists $\delta > 0$ such that for $x \in (p - \delta, p)$,

$$|f_n(x) - A_n| < \frac{\epsilon}{3}$$

. Putting these inequalties together, we see that for $x \in (p - \delta, p)$ we have

$$|f(x) - A| \leq \epsilon$$

Which implies $A = f(p-)$.

3

Let $p(x) = \sum_{k=1}^{n} a_k x^k$ be any polynomial. Then if $q(x) = (p(x) - p(-x))/2 = \sum_{k=1}^{n} b_k x^k$, we have that $b_k = a_k$ if k is odd, and $b_k = 0$ if k is even. In particular, $q(x)$ is an odd polynomial.

Now let $f : [-1, 1] \to \mathbb{R}$ be any continuous function, not necessarily odd. By the Weierstrass theorem there exists a sequence of polynomials $p_n(x)$ with $p_n(x) \to f(x)$ uniformally on $[-1, 1]$.

If $f, g : [-1, 1] \to \mathbb{R}$ are any two bounded functions, we obviously have $\sup_{x \in [-1,1]} |f(x) - g(x)| = \sup_{x \in [-1,1]} |f(-x) - g(-x)|$. In particular, $p_n(-x) \to f(-x)$ uniformally. Hence if we define $q_n(x) = (p_n(x) - p_n(-x))/2$, then (q_n) is a sequence of odd polynomials, and $q_n(x) \to (f(x) - f(-x))/2$ uniformally.

However, if f is odd, then $(f(x) - f(-x))/2 = (f(x) + f(x))/2 = f(x)$, so $q_n \to f$ uniformally.