Problem Set 8 Solutions, 18.100C, Fall 2012

November 14, 2012

For bounded functions $f, g : [a, b] \to \mathbb{R}$, we use the notation $||f|| = \sup\{|f(x)||x \in [a, b]\}$ and $d(f, g) = ||f - g||$.

1

We have $f_n \to f$ and $g_n \to g$ uniformally. We wish to show that $f_b g_n \to fg$ uniformally as well. Let $\epsilon > 0$. f and g are bounded by assumption, so pick $K \in \mathbb{R}$ with $||f||, ||g|| < K$. We may assume $K > \epsilon$. Pick a $\delta > 0$ with $\delta < \epsilon/(3K)$, and pick $N \in \mathbb{N}$ sufficiently large that for $n > N$, $||f - f_n||, ||g - g_n|| < \delta$, which is possible by Rudin 7.9. Note that for any such $n > N$, we have by Rudin 7.14

$$||f_n|| \leq ||f_n - f|| + ||f|| < \delta + K < 2K$$

And similarly $||g_n|| < 2K$. Let $x \in [a, b]$. We then have, for $n > N$,

$$|f(x)g(x) - f_n(x)g_n(x)| = |(f(x)g(x) - f(x)g_n(x)) + (f(x)g_n(x) - f_n(x)g_n(x)|$$

$$< |f(x)(g(x) - g_n(x))| + |g_n(x)(f(x) - f_n(x))|$$
$$= |f(x)| \cdot |g(x) - g_n(x)| + |g_n(x)| \cdot |f(x) - f_n(x)|$$
$$< K\delta + 2K\delta < \epsilon$$

Since this was true for any $x \in [a, b]$, we must have $||fg - f_n g_n|| < \epsilon$ for any $n > N$, which proves the result.

2

Let $\mathcal{F}$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$. We have to show that if $f \in \mathcal{F}$, then $\hat{f} \in \mathcal{F}$. We have

$$\hat{f}(0) = \frac{1}{4}f(2 \cdot 0) = \frac{1}{4}f(0) = 0$$

and

$$\hat{f}(1) = \frac{3}{4}f(2 - 1) + \frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1$$

We also need to show that $\hat{f}$ is continuous. At points $x \neq 1/2$ $\tilde{f}$ is continuous by Rudin 4.7, we just need to show that it is continuous at 1/2. Note that $\tilde{f}(1/2) = 3/4(f(0)) + 1/4 = 1/4$

Let $\epsilon > 0$. Pick $\delta > 0$ such that

$$|x - 0| < 2\delta \implies |f(x) - f(0)| = |f(x)| < \epsilon$$

and

$$|x - 1| < 2\delta \implies |f(x) - f(1)| = |f(x) - 1| < \epsilon$$

Now suppose $|x - 1/2| < \delta$. We wish to show that $|\tilde{f}(x) - \tilde{f}(1/2)| = |\tilde{f}(x) - 1/4| < \epsilon$. There are two possibilies.

If $x < 1/2$, then $|1 - 2x| < 2\delta$, and so $|f(2x) - 1| < \epsilon$. But then

$$|\tilde{f}(x) - \tilde{f}(1/2)| = |\frac{1}{4}f(2x) - \frac{1}{4}| = \frac{1}{4}|f(2x) - 1| < \frac{\epsilon}{4}$$

Similarly, if $x > 1/2$, then again $|2x - 1| < 2\delta$, and so $|f(2x - 1)| < \epsilon$. Then

$$|\tilde{f}(x) - \tilde{f}(1/2)| = |\frac{3}{4}f(2x - 1) + \frac{1}{4} - \frac{1}{4}| = \frac{3}{4}|f(2x - 1)| < \frac{3\epsilon}{4}$$

In either case $|\tilde{f}(x) - \tilde{f}(1/2)| < \epsilon$, so $\tilde{f}$ is continuous at 1/2.

Suppose $f, g \in \mathcal{F}$, and let $x \in [0, 1]$. If $x < 1/2$, we have

$$|\tilde{f}(x) - \tilde{g}(x)| = |\frac{1}{4}f(2x) - \frac{1}{4}g(2x)| = \frac{1}{4}|f(2x) - g(2x)| \leq \frac{1}{4}||f - g|| = \frac{1}{4}d(f, g)$$

Similarly, if $x \geq 1/2$, we have

$$|\tilde{f}(x) - \tilde{g}(x)| = |(\frac{3}{4}f(2x-1) - \frac{1}{4}) - (\frac{3}{4}g(2x-1) - \frac{1}{4})|$$

$$= \frac{3}{4}|f(2x-1) - g(2x-1)| \leq \frac{3}{4}||f-g|| = \frac{3}{4}d(f,g)$$

In either case $|\tilde{f}(x) - \tilde{g}(x)| \leq 3/4d(f,g)$, and so

$$d(\tilde{f}, \tilde{g}) = ||\tilde{f} - \tilde{g}|| \leq \frac{3}{4}d(f,g)$$

Now, $\mathcal{F}$ is a metric space with metric $d(\cdot,\cdot)$, and by what we have shown we can think of $\hat{}: \mathcal{F} \to \mathcal{F}$ as a function which contracts distances by at least $3/4$.

Now suppose that $\mathcal{F}$ is actually a complete metric space. Then by the Contraction Mapping Theorem, Rudin 9.23 (which we have proved on a previous homework), there would have to be a unique element $f \in \mathcal{F}$ with $\hat{f} = f$. So we just have to prove that $\mathcal{F}$ is complete.

Note that $\mathcal{F} \subset \mathcal{C} = \mathcal{C}([0,1],\mathbb{R})$, the set of all continuous functions $[0,1] \to \mathbb{R}$, and in fact the metric on $\mathcal{F}$ is the restriction of the metric on $\mathcal{C}$ By Rudin Theorem 7.15, $\mathcal{C}$ is a complete metric space. But closed subsets of complete metric spaces are themselves complete (you should check this if it isn't obvious to you), so if we can show that $\mathcal{F} \subset \mathcal{C}$ is closed then we are done.

We will show that $\mathcal{F} \subset \mathcal{C}$ is closed by showing that its complement is open. So let $f \in \mathcal{C} \setminus \mathcal{F}$. Then either $f(0) \neq 0$ or $f(1) \neq 1$. Without loss of generality suppose $f(0) \neq 0$. Let $\epsilon > 0$ such that $|f(0)| > 2\epsilon$. Then if $d(f,g) < \epsilon$, in particular $|f(0) - g(0)| < \epsilon$, and so $|g(0)| > \epsilon$, and $g \in \mathcal{F}^c$. In other words, $B_\epsilon(f) \subset \mathcal{F}^c$, and so $\mathcal{F}^c$ is open.

Another, more conceptual way to prove that $\mathcal{F}$ is closed is to show that for any $a \in [0,1]$, the map $ev_a : \mathcal{C} \to \mathbb{R}$ given by $ev_a(f) = f(a)$ is continuous. But the inverse image of a closed set under a continuous map is closed, and so $\mathcal{F} = ev_0^{-1}(0) \cap ev_1^{-1}(1)$ is also closed. Details are left to the interested reader.

3

For any $x \in [a, b]$, the sequence $f_1(x), f_2(x), \ldots$ is an alternating sequence of real numbers of decreasing norm, with the norm converging to 0. Hence by Rudin 3.43, or by a previous homework problem, the series $\sum_n f_n$ converges. Define a function (not necessarily continuous) $f : [a, b] \to \mathbb{R}$ by $f(x) := \sum_n f_n(x)$. Then the sequence of partial sums $s_n = \sum_{k=1}^n f_k$ converges pointwise to f. We wish to show that the convergence is uniform.

Let $\epsilon > 0$. We need to find an $N \in \mathbb{N}$ such for $n > N$ and any $x \in [a, b]$, $|f(x) - s_n(x)| < \epsilon$. We know that $f_k \to 0$ uniformally. So let $N \in \mathbb{N}$ be sufficiently large that $|f_k(x)| < \epsilon$ for all $k > N$, $x \in [a, b]$.

We now need the following

Lemma: Suppose (a_n) is an alternating sequence as in Rudin 3.43, and $a = \sum_n a_n$. Then $|a| < |a_1|$.

Assume the Lemma for the moment. For any $n > N$, indeed any n, we have

$$f(x) - s_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$$

But $\sum_{k=n+1}^{\infty} f_k(x)$ is itself an alternating series. By the Lemma, we then have

$$|f(x) - s_n(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| < |f_{n+1}(x)| < \epsilon$$

Since the choice of N did not depend on the point x, we have $s_n \to f$ uniformally.

Proof of Lemma: We first show that a_1 and a have the same sign. We have

$$a = \sum_{n=1}^{\infty} a_n = (a_1 + a_2) + (a_3 + a_4) + \cdots = \sum_{n=1}^{\infty} (a_{2n-1} + a_{2n})$$

All terms $a_{2n-1} - a_{2n}$ in the sum on the right have the same sign as a_1, and so a must also have the same sign as a_1.

Now assume $a_1 > 0$. Then $a > 0$ as well, and $a - a_1 = \sum_{n=2}^{\infty} a_n$. But by the previous paragraph, the latter sum has the same sign as a_2, which is negative. Hence $a - a_1 < 0$, and $0 < a < a_1$, so $|a| < |a_1|$. The result follows for $a_1 < 0$ by replacing a_n with $-a_n$.