Problem Set 5 Solutions, 18.100C, Fall 2012

October 18, 2012

1

Let $s_n = \sum_{i=1}^n x_i$ and $\sigma_n = \sum_{i=1}^n y_i$ be the partial sums. Then we claim that σ_n is the average of the first n s_n, i.e.

$$\sigma_n = \frac{s_1 + s_2 + \cdots s_n}{n}$$

To see this, we will look at the "coefficients" of the x_k in the expressions for the y_j; if this makes you nervous, think of them as variables, for which we will later plug in values to get actual sequences of real numbers. If $k > j$, then x_k does not appear in the expression for y_j, while if $k \leq j$ then x_k appears with a coefficient of $(k-1)/(j(j-1))$, where this expression should be interpreted as equal to 1 in the degenerate case $k = j = 1$. Note that $1/(j(j-1)) = 1/(j-1) - 1/j$. Using this, we can determine the coefficient of x_k in σ_n; this is zero if $j > n$, and if $j \leq n$, it is equal to

$$\sum_{j=k}^{n} \frac{k-1}{j(j-1)} = (k-1)\sum_{j=k}^{n}(\frac{1}{j-1} - \frac{1}{j}) = (k-1)(\frac{1}{k-1} - \frac{1}{n}) = \frac{n-k+1}{n}$$

Where we got rid of the sum by noting that the negative part of one term of the sum is equal to the positive part of the next term, and hence the sum "telescopes" to $1/(k-1) - 1/n$. But now consider the sum of partial sums $s_1 + s_2 + \cdot + s_n$; x_k will appear precisely $n - k + 1$ times as long as $k \leq n$, and hence the coefficient of x_k in $(s_1 + s_2 + \cdots + s_n)/n$ is $(n - k + 1)/n$, which is exactly the coefficient we computed above for σ_n, and hence σ_n is the average of the s_n as desired.

Now, by assumption there is a number s with $s = \sum_{i=1}^{\infty} x_i$. More precisely, this means that $s_n \to s$ as $n \to \infty$. Hence the fact that $s = \sum_{i=1}^{\infty} y_i$ will follow from the following

Fact: Suppose $\{s_n\}$ is a sequence of real numbers with $s_n \to s$ as $n \to \infty$. Then if $\sigma_n = \frac{s_1+s_2+\cdots s_n}{n}$, then $\sigma_n \to \infty$ as $n \to \infty$

Proof: The idea is to split the sum determining σ_n into two parts; one part will be small because the denominator is large, and the other part will be close to s. Fix $\epsilon > 0$, let $N \in \mathbb{N}$ be sufficiently large that for $n > N$ we have $|s - s_n| < \epsilon$. With this N fixed, choose $M >> N$ such that $(|\sum_{i=1}^{N} s_i|)/M < \epsilon$ and such that $N|s|/M < \epsilon$. Note that both these inequalities will continue to hold for $n > M$. For any such n, we compute

$$|\sigma_n - s| = \frac{|\sum_{i=1}^{N} s_i + \sum_{i=N+1}^{n} s_i - ns|}{n}$$
$$\leq \frac{|\sum_{i=1}^{N} s_i|}{n} + \frac{|\sum_{i=N+1}^{n} s_i - (n-N)s|}{n} + \frac{N|s|}{n}$$
$$< \epsilon + \frac{|\sum_{i=N+1}^{n} |s_i - s|}{n} + \epsilon < 2\epsilon + \frac{(n-N)\epsilon}{n} < 3\epsilon$$

Thus $\sigma_n \to s$ as $n \to \infty$ and the claim is proved.

Now take the alternating series $1 - 1 + 1 - 1 + \cdots$. This has partial sums $s_i = 1$ if i is odd, and 0 if i is even. Thus taking averages, we have $\sigma_n = 1/2$ if n is even, and $(n+1)/(2n)$ if n is odd. Since $\lim_{n\to\infty}(n+1)/(2n) = 1/2$, we see that $\sigma_n \to 1/2$.

2

We define $f(x) = 1/2(x + \alpha/x)$ for $x \in \mathbb{R}$. We have

Claim: suppose $x > \sqrt{\alpha}$. Then $x > f(x) > \sqrt{\alpha}$.

Proof: We have

$$x - f(x) = x - 1/2(x - \alpha/x) = 1/2(x - \alpha/x) = 1/2((x^2 - \alpha)/x) > 0$$

Since $x^2 > \alpha > 0$. For the other inequality, we will show that $f(x)^2 > \alpha$. We compute

$$(\frac{1}{2}(x + \frac{\alpha}{x})^2 = \frac{\alpha}{2} + \frac{1}{4}(x^2 + \frac{\alpha^2}{x^2}) > \frac{\alpha}{2} + \frac{1}{2}\sqrt{\frac{x^2\alpha^2}{x^2}} = \alpha$$

We used the AM-GM inequality, which states that for any positive real numbers a and b, $(a+b)/2 > \sqrt{ab}$, with equality if and only if $a = b$; this can be proved by noting that $(\sqrt{a} - \sqrt{b})^2 \geq 0$, with equality if and only if $a = b$. This proves the claim.

Now, we start with some $x_1 > \alpha$, and we inductively define $x_{n+1} = f(x)$. Then by the claim, (x_n) is a decreasing sequence, bounded from below by $\sqrt{\alpha}$. By Rudin Theorem 3.14, this sequence converges to some x. Since x is the inf of (x_i), we must have $x \geq \sqrt{\alpha} > 0$. Hence by Rudin Theorem 3.3, we have $\lim_{n\to\infty} 1/x_n = 1/x$. Applying this Theorem repeatedly, we then have $\alpha/x_n \to \alpha/x$, then $x_n + \alpha/x_n \to x + \alpha/x$, and finally $1/2(x_n + \alpha/x_n) \to 1/2(x + \alpha/x)$. In other words, we have $f(x_n) \to f(x)$ as $x \to \infty$ (readers who are familiar with the concept will note that this argument amounts to proving the continuity of f). But by the definition of (x_n), we have

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} x_{n+1} = x$$

In other words, $x = 1/2(x + \alpha/x)$, or $x^2 = \alpha$. This means that $x = \pm\sqrt{\alpha}$, but since $x \geq \sqrt{\alpha}$, we have $x = \sqrt{\alpha}$.

3

We have an alternating sequence (c_i), i.e. $c_{2k} > 0$ and $c_{2k+1} < 0$ for all $k \in \mathbb{N}$, with the property that $|c_k| > |c_{k+1}|$ and $|c_k| \to 0$ as $k \to \infty$. We have the partial sums $s_n = \sum_{k=1}^{n}$ of c_k, and we wish to show that (s_n) converges. Suppose $n, m \in \mathbb{N}$, and that $n > m$. When comparing s_n and s_m, there are four cases, depending on the parity of m and n

Case 1: n and m are odd. Then $s_n > s_m$. Indeed, note that $s_{m+2} - s_m = c_{m+1} + c_{m+2}$. Since $m+1$ is even, $c_{m+1} > 0$, and $c_{m+1} > |c_{m+2}|$, so $c_{m+1} + c_{m+2} > 0$.

Case 2: n and m are even. Then $s_n < s_m$. The argument is the same as Case 1.

Case 3: n is odd and m is even. Then $s_m > s_n$. Indeed, by Case 2 $s_m > s_{n+1}$. But $s_{n+1} - s_n = c_{n+1} > 0$.

Case 4: n is even and m is odd. Then $s_n > s_n$. Same argument as Case 3.

Putting these together, we see that $s_1 < s_3 < \cdots$ is an increasing sequence, bounded above, $s_2 > s_4 > \cdots$ is a decreasing sequence, bounded from below, and that if k is even and j is odd then $s_k > s_j$. Thus by Rudin Theorem 3.14 there are real numbers r, s with $\lim_{k\to\infty} s_{2k} = s$ and $\lim_{k\to\infty} s_{2k+1} = r$.

If one examines the proof in Rudin, one sees that r is the least upper bound of the (s_{2k+1}), and s the greatest lower bound of (s_{2k}). For any $n \in \mathbb{N}$, s_{2n+1} is a lower bound for the set of even s_k, and hence $s_{2n+1} \leq s$. Since r is the sup of the odd ones, this means that $r \leq s$.

If $r = s$, then we are done, since for any $\epsilon > 0$, take N sufficiently large such that for for any $k > N$, then both $|s - s_{2k}| < \epsilon$ and $|s_{2k+1}| < \epsilon$; then $2N$ will work for this ϵ and $s_n \to s$.

So suppose $s > r$. Pick $\delta < (r-s)/2$ and k odd and sufficiently large that $\delta > c_{k+1} > 0$. Then $s_k \leq r$ and $s_{k+1} \geq s$ and so

$$c_{k+1} = s_{k+1} - s_k \geq s - r > \delta > c_{k+1}$$

Which is a contradiction.

4

We are looking for an explicit rearrangment of the convergent series $1 - 1/2 + 1/3 - 1/4 + \cdots$ that does not converge. We will construct a rearrangment whose partial sums go to infinity as $n \to \infty$.

Let $x_k = 1/(2k+1)$ and $s_n = \sum_{k=0}^{n} x_n$. Then (s_n) is an increasing sequence, and I claim $s_n \to \infty$ as $n \to \infty$. In fact, we can be more precise:

$$2s_n = \sum_{k=0}^{n} \frac{1}{2k+1} + \sum_{k=0}^{n} \frac{1}{2k+1} > \sum_{k=0}^{n} \frac{1}{2k+1} + \sum_{k=0}^{n} \frac{1}{2k+2}$$

$$= \sum_{j=1}^{2n+2} \frac{1}{j} = s'_{2n+2}$$

Where s'_{2n+2} is the partial sum of the harmonic series. But $\sum_{j=1}^{\infty} 1/j = \infty$, so s_n must also diverge.

Now we pick an increasing sequence $N_1, N_2 \ldots$ of positive integers as follows. Set $N_1 = 1$. Having chosen $N_1, \ldots N_k$, we pick an N_{k+1} such that $s_{N_{k+1}} - s_{N_k} > 1/2$; since $s_n \to \infty$, this is always possible.

In fact, we can be a little more precise in our choice of N_k. An slight refinement of the above argument shows that, for $n > m$, we have $2(s_n - s_m) > s'_{2n+1} - s_{2m}$. But the proof of the divergence of the harmonic series shows that $s'_{2^{k+1}} - s'_{2^k} > 1/2$; indeed, this sum has 2^k terms in it, each of which is larger than $1/2^{k+1}$. Thus if we take $N_k = 2^{2(k-1)}$, we will have

$$s_{N_{k+1}} - s_{N_k} > \frac{1}{2}(s_{2^{2k+1}+1} - s_{2^{2k-1}} > \frac{1}{2}$$

as desired.

We can now desribe our divergent rearrangment of the sum:

$$1 - \frac{1}{2} + \sum_{j=N_1+1}^{N_2} \frac{1}{2j+1} - \frac{1}{4} + \cdots + \sum_{j=N_{k-1}+1}^{N_k} \frac{1}{2j+1} - \frac{1}{2k} + \cdots$$

By construction, the negative terms of this sum are very sparse, occuring only at numbers of the form $N_k + k$; for notational convenience, we define $M_k = N_k + k$. Let t_n be the n'th partial sum; our goal is to prove $\lim_{n\to\infty} t_n = \infty$.

Fact 1: if $M_k < n < M_{k+1}$, then $t_{M_k} < t_n$

Proof: $t_n - t_{M_k} = \sum_{j=N_k+1}^{n-k} 1/(2j+1) > 0$

Fact 2: for $k \geq 1$, $t_{M_{k+1}} - t_{M_k} > 1/4$

Proof:

$$t_{M_{k+1}} - t_{M_k} = \sum_{j=N_k+1}^{N_{k+1}} 1/(2j+1) - 1/(2k+2) > 1/2 - 1/(2k+2) > 1/4$$

Fact 3: $t_{M_k} > k/2$

Proof: $t_{M_1} = 1/2$, so the result follows from Fact 2 and induction.

With these facts, we can show that t_n diverges. Let r be any real number, and pick $k \in \mathbb{N}$ such that $k/2 > r$. Then I claim that for $n > M_k$, $t_n > k/2 > r$. To see this, for any such n take k' such that $M_{k'} \leq n < M_{k'+1}$; then by fact 1 $t_n \geq t_{M_{k'}}$. We must have $k' \geq k$, since $n > M_k$, and so by fact 3 $t_{M_{k'}} > k'/2 \geq k/2 > r$. This proves that $t_k \to \infty$ as $k \to \infty$.

18.100C Real Analysis
Fall 2012