Problem Set 4 Solutions, 18.100C, Fall 2012

October 11, 2012

1

Let X be a complete metric space with metric d, and let $f : X \to X$ be a contraction, meaning that there exists $\lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Then there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof:

Existence: Let $x_1 \in X$ be arbitary and inductively let $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. We will prove that (x_n) is a Cauchy sequence. Suppose inductively that

$$d(x_{r+1}, x_r) \leq \lambda^{r-1} d(x_2, x_1).$$

Then

$$d(x_{r+2}, x_{r+1}) = d(f(x_{r+1}), f(x_r)) \leq \lambda d(x_{r+1}, x_r) \leq \lambda^r d(x_2, x_1)$$

so that the above equation holds for all $r \in \mathbb{N}$. For $m > n$, by repeated use of the triangle inequality

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n).$$

Hence,

$$d(x_m, x_n) \leq (\lambda^{m-2} + \ldots \lambda^{n-1}) d(x_2, x_1) = \frac{\lambda^{n-1}(1 - \lambda^{m-n})}{1 - \lambda} d(x_2, x_1) \leq \frac{\lambda^{n-1}}{1 - \lambda} d(x_2, x_1).$$

Let $\epsilon > 0$. By theorem $3.20(e)$ and $3.3(b)$, there exists an $N \in \mathbb{N}$ such that

$$n \geq N \implies \lambda^{n-1} d(x_2, x_1) < \epsilon(1 - \lambda)$$

and so

$$m, n \geq N \implies d(x_m, x_n) < \epsilon,$$

which shows (x_n) is Cauchy. Since X is complete (x_n) converges to some $x_0 \in X$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_0, x_n) < \frac{\epsilon}{2}$$

and so

$$d(x_0, f(x_0)) \leq d(x_0, x_{N+1}) + d(f(x_N), f(x_0)) \leq d(x_0, x_{N+1}) + \lambda d(x_N, x_0) < \epsilon.$$

Since ϵ was arbitary, $d(x_0, f(x_0)) = 0$ giving $x_0 = f(x_0)$, as required.

Uniqueness: If $f(x_0) = x_0$ and $f(y_0) = y_0$ then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \lambda d(x_0, y_0) \implies (1 - \lambda) d(x_0, y_0) \leq$$
$$0 \implies d(x_0, y_0) \leq 0.$$

Thus $d(x_0, y_0) = 0$ giving $x_0 = y_0$.

2

We have a convergent sequence $x_k \to x$, and a bijective function $g : \mathbb{N} \to \mathbb{N}$, with an inverse function g^{-1}, and we wish to show that x'_k also converges to x. Pick $\epsilon > 0$; we will find N' such that $k' > N' \implies d(x, x_{k'}) < \epsilon$

Since $x_k \to x$, there exists $N \in \mathbb{N}$ such that $k > N \implies d(x, x_k) < \epsilon$. Now, pick an $N' \in \mathbb{N}$ greater than $\max\{g^{-1}(1), g^{-1}(2) \ldots g^{-1}(N)\}$, which is always possible since this is a finite set. Let $k' > N'$, and consider $x'_k = x_{g(k')}$. Let $k = g(k')$; then we must have $k > N$. If not, then $k \leq N$ and by the definition of N' we have $k' = g^{-1}(k) < N' < k'$, a contradiction. So we have $d(x, x'_k) = d(x, x_{g(k)}) < \epsilon$. So this N' works, and we are done.

This statement is no longer true if g is not one-to-one. As a counter-example, consider the sequence of real numbers $x_k = 1/k$, and the function $g : \mathbb{N} \to \mathbb{N}$ give by $g(n) = 1$ if n is odd, $g(n) = 2$ if n is even. Then $x_k \to 0$, but x'_k simply alternates between 1 and 1/2, and hence is not a Cauchy sequence and cannot converge.

3

To set notation, we use boldface for vectors in $\mathbb{R}^n$, i.e. $\mathbf{x} \in \mathbb{R}^n$, and superscripts with the same letter non-boldfaced for components of that vector, i.e. x^j is the j'th component of $\mathbf{x}$, $1 \leq j \leq n$. We use lower subscripts for sequences; $\{\mathbf{x}_i\}$ will be a sequence in $\mathbb{R}^n$, and x_i^j is the j'th component of the i'th vector in the sequence.

With that set, we can proceed with the problem. Suppose first that $\mathbf{x}_i \to \mathbf{x}$; we need to show that $x_i^j \to x^j$ as $i \to \infty$. Note first that, for any two vectors $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, we have $(y^j - z^j)^2 \leq \sum_{l=1}^n (y^l - z^l)^2$. Taking the positive square root of both sides, we see that $|y^j - z^j| \leq ||\mathbf{y} - \mathbf{z}||$. Now, take any $\epsilon > 0$, and N sufficiently large that for $k > N$, $||\mathbf{x} - \mathbf{x_k}|| < \epsilon$. Then by what we just showed $|x^j - x_k^j| < \epsilon$, so this N also works for ϵ and the sequence $\{x_k^j\}$, so we have $x_k^j \to x^j$ as desired.

For the other direction, suppose that x_k^j converges to some real number x^j for each $1 \leq j \leq n$. Then take the vector $\mathbf{x}$ whose j'th component is x^j. We will show that $\mathbf{x}_k \to \mathbf{x}$. Fix $\epsilon > 0$. Since $x_k^j \to x^j$, we can choose for each j a natural number N^j such that for $k > N^j$, $|x^j - x_k^j| < \epsilon/\sqrt{n}$; recall that n here is the dimension $\mathbb{R}^n$. Now take N bigger than any of the $N^1, N^2, \ldots N^n$; we claim that for $k > N$, $||\mathbf{x} - \mathbf{x}_k|| < \epsilon$, so that this N works for this choice of ϵ, and we have shown that $\mathbf{x}_k \to \mathbf{x}$. We compute

$$||\mathbf{x}-\mathbf{x}_k|| = \sqrt{(x^1 - x_k^1)^2 + (x^2 - x_k^2)^2 + \cdots + (x^n - x_k^n)^2} < \sqrt{(\epsilon/\sqrt{n})^2 + \cdots + (\epsilon/\sqrt{n})^2}$$

$$= \sqrt{n\epsilon^2/n} = \epsilon$$

Where we used the fact that the square root function is increasing in the second step. This completes the proof.

4

Recall that the p-adic metric is defined as follows: if $a, b \in \mathbb{Z}$, let n be the largest power of p that divides $a - b$, i.e. $p^n | (a - b)$, but $p^{n+1} \nmid (a - b)$. Then $d(a, b) = 1/p^n$.

Now we wish to show that the sequnce $x_k = \sum_{i=0}^{k-1} p^i$ is Cauchy. Note that p does not divide any x_k; indeed, $x_k - 1$ is divisible by p, and no consequetive numbers are divisible by p. Consider any pair x_n, x_m, where without loss of generality $n > m$. Then

$$x_n - x_m = \sum_{i=0}^{n-1} p^i - (\sum_{i=0}^{m-1} p^i) = \sum_{i=m}^{n-1} p^i = p^m(\sum_{i=0}^{n-m-1} p^i) = p^m x_{n-m}$$

Since p does not divide x_{n-m}, this means that the largest power of p dividing $x_n - x_m$ is m. In other words, $d(x_n, x_m) = 1/p^m$. This formula shows that $\{x_k\}$ is Cauchy. Indeed, let $\epsilon > 0$. By Rudin 3.20 (e), we can find $N \in \mathbb{N}$ sufficiently large that $1/p^N < \epsilon$. Then if $n > m > N$, we have $d(x_n, x_m) = 1/p^m < 1/p^N < \epsilon$, and so the sequence is Cauchy.

Now consider the case $p = 2$. By the formula for the sum of a geometric series we have

$$x_n = \sum_{i=0}^{n-1} 2^i = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

So $x_n - (-1) = 2^n$, and has n the highest power of 2 dividing it. This says that $d(x^n, -1) = 1/2^n$. But since the numbers $1/2^n \to 0$ as $n \to \infty$, this shows that $x_n \to 1$, so the sequence converges.