**SOLUTIONS TO PS 8**

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**Solution/Proof of Problem 1.** *From* $f(x) = f(x^2)$, *we have*

$$f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = \cdots = f(x^{\frac{1}{2n}}).$$

*Now let* $y_n = x^{\frac{1}{2n}}$, *and assume* $x \neq 0$, *so* $\lim_{n\to\infty} y_n = 1$. *Since* $f$ *is continuous, we have if* $x \neq 0$

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty} y_n) = f(1).$$

*When* $x = 0$, *then* $f(0) = \lim_{y\to 0} f(y) = f(1)$.

*Then* $f$ *is a constant.*

**Solution/Proof of Problem 2.** *From MVT, we have* $\forall x > 0, \exists y = y(x) \in (x, x+1)$, *s.t.*

$$g(x) = f(x+1) - f(x) = f'(y).$$

*Notice that* $y > x$, *so* $\lim_{x\to\infty} y = \infty$, *so we have*

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(y) = 0,$$

*since* $\lim_{y\to\infty} f'(y) = 0$.

**Solution/Proof of Problem 3.** *Consider the function*

$$g(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_n x^{n+1}}{n+1}.$$

*Then* $g(0) = 0$ *and* $g(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$. *By the mean value theorem, we have that* $g'(y) = 0$ *for some* $y \in (0,1)$, *which means the equation* $C_0 + C_1 x + \cdots + C_n x^n$ *has a root between* $0$ *and* $1$.

**Solution/Proof of Problem 4.** *(a) Suppose* $f$ *has two fixed point,* $x_1 < x_2$. *Then by MVT we have that*

$$\exists y \in (x_1, x_2), \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(y).$$

*Then* $f'(y) = 1$ *which is a contradiction.*

*(b) If* $f$ *has a fixed point, then*

$$f(t) = t \;\Rightarrow\; t = t + (1 + e^t)^{-1} \;\Rightarrow\; (1 + e^t)^{-1} = 0.$$

*But we know that* $e^t > 0$ *then* $(1 + e^t)^{-1} \neq 0$. *So* $f$ *has no fixed point.*

*(c)Consider a sequence defined by* $x_n = f(x_{n-1})$ *for any* $x_1 \in \mathbb{R}$. *Then we have*

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| = |f'(y)||x_{n-1} - x_{n-2}| \leqslant A|x_{n-1} - x_{n-2}|.$$

*By using* $|x - z| \leqslant |x - y| + |y - z|$, *we have*

$$|x_n - x_m| \leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \leqslant \left( \sum_{i=m-1}^{n-2} A^i \right) |x_2 - x_1|.$$

*Since $A < 1$, the series $\sum A^i$ converges, and so the partial sums form a Cauchy sequence. This inequality shows that $\{x_n\}$ is also a Cauchy sequence and hence converges.*

*Notice that $|f(x+h) - f(x)| \leqslant Ah$, so $f$ is continuous. Then we have*

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x).$$

*So $x$ is a fixed point of $f(x)$.*

**Solution/Proof of Problem 5.** *$f'(0)$ exists because of the following:*

*By definition, $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x}$. Since we have $\frac{(f(x)-f(0))'}{(x)'} = f'(x)$, and the limit $\lim\limits_{x\to 0} f'(x) = 3$ exists, from L'Hospital rule, we know $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x}$ exists and $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = 3$.*

**Solution/Proof of Problem 6.** *From Taylor's theorem, we have*

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f^{(2)}(x_0)(x-x_0)^2 \\ &\quad + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x-x_0)^{n-1} + \frac{1}{n!} f^{(n)}(y)(x-x_0)^n, \end{aligned}$$

*for some $y \in (x, x_0)$ or $(x_0, x)$. Then we have*

$$f(x) = f(x_0) + \frac{1}{n!} f^{(n)}(y)(x-x_0)^n \quad \Rightarrow \quad f^{(n)}(y) = n! \frac{f(x)-f(x_0)}{(x-x_0)^n}.$$

*We want to say that $f^{(n)}(y)$ has the same sign as $f^{(n)}(x_0)$, but we have to be careful because $f^{(n)}$ need not be continuous. It may well be the case that there is a sequence $z_n$ approaching $x_0$ such that $f^{(n)}(z_n)$ does <u>not</u> go to $f^{(n)}(x_0)$. Nevertheless, we will show that if $y(x)$ is the intermediate point in $(x, x_0)$ appearing in Taylors's theorem, then as $x \to x_0$, $f^{(n)}(y) \to f^{(n)}(x_0)$.*

*The reason is that, as pointed out above,*

$$\lim_{x\to x_0} f^{(n)}(y) = \lim_{x\to x_0} n! \frac{f(x)-f(x_0)}{(x-x_0)^n},$$

*and we can compute the limit on the right by using L'Hôpital's rule $n-1$ times:*

$$\lim_{x\to x_0} n! \frac{f(x)-f(x_0)}{(x-x_0)^n} = \lim_{x\to x_0} (n-1)! \frac{f'(x)}{(x-x_0)^{n-1}} = \ldots = \lim_{x\to x_0} \frac{f^{(n-1)}(x)}{(x-x_0)}$$

*and then noticing that since $f^{(n-1)}(x_0) = 0$, this is equal to*

$$\lim_{x\to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{(x-x_0)} = f^{(n-1)}(x_0) = A$$

*If $A > 0$, then $\frac{f(x)-f(x_0)}{(x-x_0)^n} > 0$ in a neighborhood of $x_0$. If $n$ even, this implies $f(x) - f(x_0) > 0$ for any element in the neighborhood of $x_0$, i.e $x_0$ is a local minimum. When $n$ is odd, $f$ does not have a local minimum or maximum.*

*Similarly, if $A < 0$ and $n$ even, then $f(x) - f(x_0) < 0$ for any element in the neighborhood of $x_0$, i.e $x_0$ is a local maximum. When $n$ is odd, $f$ does not have a local minimum or maximum.*

**Solution/Proof of Problem 7.** *For $x > 0$ we have $f(x) = x^3$ and hence $f'(x) = 3x^2$ and $f''(x) = 6x$. For $x < 0$ we have $f(x) = -x^3$ and hence $f'(x) = -3x^2$ and $f''(x) = -6x$. Notice that we can write, for $x \neq 0$,*

$$f(x) = |x|x^2, \quad f'(x) = 3|x|x, \quad f''(x) = 6|x|.$$

*Hence at $x = 0$ we have*

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|h^2}{h} = \lim_{h\to 0} |h|h = 0$$

$$f''(0) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0} \frac{3|h|h}{h} = \lim_{h\to 0} 3|h| = 0$$

$$f'''(0) = \lim_{h\to 0} \frac{f''(h) - f''(0)}{h} = \lim_{h\to 0} \frac{6|h|}{h} \text{ does not exist}$$

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