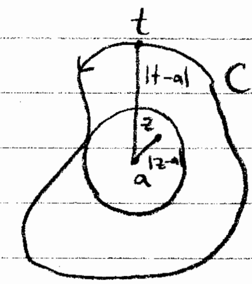
Taylor Series



$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z}\,dt$

- t moves along C.
- Want to find a series expansion in powers $(z-a)^n \quad n=0,1,2,\ldots$

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} \qquad |t-a|>|z-a|$$

$$= \frac{1}{t-a}\,\frac{1}{1-\underbrace{\frac{z-a}{t-a}}_{\lambda}} \qquad |\lambda|<1 \longrightarrow$$

$$\boxed{\underbrace{\sum_{n=0}^{\infty}\lambda^n}_{\text{geometric series}} = \frac{1}{1-\lambda},\ |\lambda|<1,\ \lambda:\text{complex}}$$

$$= \frac{1}{t-a}(1+\lambda+\ldots+\lambda^n+\ldots) = \frac{1}{t-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{t-a}\right)^n = \frac{1}{t-z}$$

$$\therefore f(z) = \frac{1}{2\pi i}\oint_C\left[f(t)\frac{1}{t-a}\sum_{n=0}^{\infty}\frac{(z-a)^n}{(t-a)^n}\right]dt$$

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n\oint_C\frac{f(t)}{(t-a)^{n+1}}\,dt = \sum_{n=0}^{\infty}A_n(z-a)^n$$

$$f(z) = A_0 + A_1(z-a) + A_2(z-a)^2 + \ldots + A_n(z-a)^n + \ldots$$

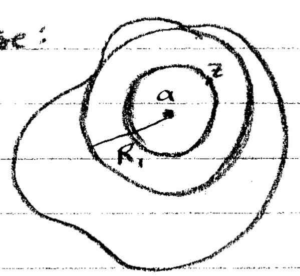
$$f(a) = A_0\ ,\quad f'(a) = A_1,\quad f''(a) = 2A_2\ \ldots\quad f^{(n)}(a) = n!\,A_n$$

$$\boxed{f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n}$$

$$\boxed{\text{Taylor Series at } a:\quad f^{(n)}(a) = n!\,\frac{1}{2\pi i}\oint_C\frac{f(t)}{(t-a)^{n+1}}\,dt \qquad n=0,1,2,\ldots}$$

Where does the Taylor series converge for fixed a?

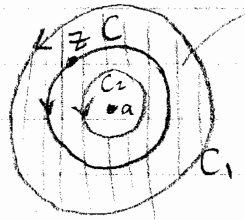
Suppose:



$|z-a| < |\cancel{t-a}|$

$0 \le |z-a| < R_1$

Laurent Series - generalization of Taylor series



f(z): analytic

← can move C_1, C_2 to any C and the integrand won't change

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\left[\oint_{C_2}\frac{f(t)}{t-z}\,dt - \oint_{C_1}\frac{f(t)}{t-z}\,dt\right]$$

· t on C_2: $\dfrac{1}{t-z} = \dfrac{1}{(t-a)-(z-a)} = \displaystyle\sum_{n=0}^{\infty}\frac{(z-a)^n}{(t-a)^{n+1}}$ ①

$(|t-a|>|z-a|)$

· t on C_1: $\dfrac{1}{t-z} = \dfrac{1}{(t-a)-(z-a)} = \dfrac{-1}{(z-a)}\,\dfrac{1}{1-\frac{t-a}{z-a}} = -\dfrac{1}{z-a}\displaystyle\sum_{n=0}^{\infty}\frac{(t-a)^n}{(z-a)^n}$

$$= \sum_{n=0}^{\infty}\frac{(t-a)^n}{(z-a)^{n+1}} \quad ②$$

① $\dfrac{1}{2\pi i}\oint_{C_1}\dfrac{f(t)}{t-z}\,dt = \displaystyle\sum_{n=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_2}\frac{f(t)}{(t-a)^{n+1}}\,dt\right]}_{A_n = D_n,\ n\geq 0}(z-a)^n$

② $-\dfrac{1}{2\pi i}\oint_{C_1}\dfrac{f(t)}{t-z}\,dt = \displaystyle\sum_{\substack{n=0\\ m=-\infty}}^{\substack{(m:)-\infty\\ \infty}}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_1} f(t)(t-a)^n\,dt\right]}_{D_m}(z-a)^{\overset{m}{-n-1}}$

$m = -n-1$

$n = -m-1$

$$\boxed{f(z) = \sum_{m=-\infty}^{\infty} D_m (z-a)^m, \qquad D_m = \frac{1}{2\pi i}\oint_c \frac{f(t)}{(t-a)^{m+1}}\,dt}$$ independent of z

c in annulus.

$\rho_1 < |z-a| < \rho_2$

smallest radius (ρ_1), largest radius (ρ_2)

<u>ex</u> $f(z) = e^z/z^k$ $k = 1, 2, \ldots$ find Laurent series $(a=0)$

$$e^z = 1 + z + z^2/2 + \ldots + z^n/n! + \ldots$$

$$\frac{e^z}{z^k} = \frac{1}{z^k} + \frac{1}{z^{k-1}} + \frac{z^{2-k}}{2!} + \frac{z^{n-k}}{n!} + \ldots$$

↑ negative power of z

converges for $0 < |z - \overset{0}{\cancel{a}}| < \infty$