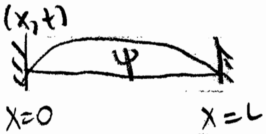
lambda takes discrete values

Fourier Series

$\Psi = \Psi(x,t)$



$$\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

boundary condition: $\Psi|_{x=0} = 0 = \Psi|_{x=L}$

initial condition: $\Psi|_{t=0} = f(x)$: known $\qquad \frac{\partial\Psi}{\partial t}\Big|_{t=0} = 0$

<u>Separation of variables</u>: $\Psi(x,t) = X(x)T(t)$

PDE: $X''(x)T = \frac{1}{c^2}XT'' \leftrightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \text{const} = -k^2$

if k is positive: $T'' - (k^2c^2)T = 0$

$\leftrightarrow T(t) = \cancel{Ae^{kct}} + Be^{-kct}$

blows up as $t \to \infty$

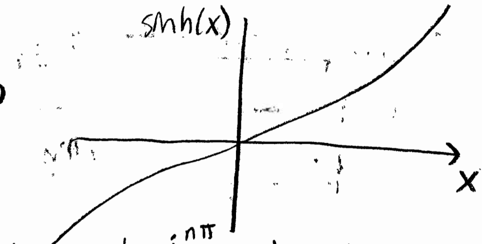
$X: X'' - k^2X = 0$

$\leftrightarrow X(x) = Ce^{kx} + De^{-kx}$

$= C\sinh(kx) + D\cosh(kx)$

Boundary conditions: $X(x=0)=0 \to D=0$

$X(x=L)=0 \to \sinh(kL)=0$



That would mean $k=0$. unless...

let $z=kL$

$\sinh(z)=0 \to z = in\pi, \quad n=0,\pm1,\pm2, \text{etc} \to k = i\frac{n\pi}{L}$: imaginary

$k^2=-q^2<0 \to \boxed{q=q_n=\frac{n\pi}{L}}$ $\quad n\neq 0$

$q=-ik$ $\quad n=1,2,3,\ldots$ (or negative)

$$\boxed{X(x)=\tilde{C}\sin\left(\frac{n\pi x}{L}\right)}$$

$k=iq \to \boxed{T(t)=\tilde{A}\cos(qct)+\tilde{B}\sin(qct)}$ $\quad$ ODE for $T(t)$: $T''+(q^2c^2)T=0$

2nd initial condition: $\left.\frac{\partial\psi}{\partial t}\right|_{t=0}=0 \to T'(0)=0$

$$\left.T'(t)\right|_{t=0}=-qc\tilde{A}\sin(qct)+qc\tilde{B}\cos(qct)=qc\tilde{B}=0 \to \tilde{B}=0$$

$$\boxed{T(t)=\tilde{A}\cos(qct)}$$

$$\psi(x,t)=X(x)\cdot T(t)=\underbrace{\tilde{A}\tilde{C}}_{E}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)$$

$\psi(x,t=0)=E\sin\left(\frac{n\pi x}{L}\right)\neq f(x)$ $\quad$ unless $f(x)=\text{const.}\sin\left(\frac{n\pi x}{L}\right)$

<u>Digression</u>: If PDE: $\underset{\text{operator}}{\mathcal{L}}\psi=0$ and $\mathcal{L}$: linear, then if ψ_1, ψ_2 are solutions of the PDE, $\psi_1+\psi_2$ is also a solution.

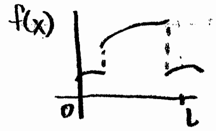
$\mathcal{L}$: linear iff $\mathcal{L}(y_1+y_2)=\mathcal{L}(y_1)+\mathcal{L}(y_2)$

<u>Fourier's Proposal</u>: Seek a solution

$$\psi(x,t)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)=0$$

initial condition: $\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{L}\right)=f(x) \longrightarrow$ find E_n

Fourier proved that for any piecewise continuous function $f(x)$, this is true:



piecewise continuous

Any piecewise continuous $f(x)$ can be expanded in sines.

Convergence is understood as "convergence in the mean"

$\therefore \lim_{N\to\infty} \int_0^L dx \left| f(x) - \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\right|^2 = 0$

orthogonality of sines:

$$\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & n \neq m \\ L/2, & n = m \end{cases}$$

$n=m$: $\quad \sin^2\left(\frac{n\pi x}{L}\right) = \dfrac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2} \to \frac{1}{2}\int_0^L dx \left(1-\cos\left(\frac{2n\pi x}{L}\right)\right) = \frac{L}{2}$

$$\int_0^L dx \underbrace{\sum_{n=1}^{\infty} E_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}_{0,\ n\neq m} = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$n=m$: $\quad E_m \frac{L}{2} = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad \boxed{E_m = \frac{2}{L}\int_0^L dx\, f(x)\sin\left(\frac{m\pi x}{L}\right)}$

Theme: consider the Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

$y=y(x)$, homogeneous boundary conditions, e.g. $y(a)=0=y(b)$, $y'(a)=0=y'(b)$

This is a "proper" SL problem.

Then,

- λ is in $\{\lambda_n\}_{n=1}^{\infty}$ eigenvalues with characteristic function $y = c\,\varphi_n(x)$ satisfies the SL problem with $\lambda = \lambda_n$
- $\int_a^b r(x)\varphi_n(x)\varphi_m(x)dx = 0 \quad \lambda_n \neq \lambda_m$ (orthogonality)
- for any "admissible" function $f(x)$, we can write

$$f(x) = \sum_n c_n \varphi_n(x)$$

where

$$c_n = \frac{\int_a^b r(x) f(x)\varphi_n(x)dx}{\int_a^b r(x)\varphi_n^2(x)dx}$$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$