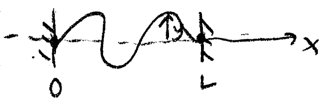
Boundary-Value Problems

A. Vibration of a String



$y = y(x,t)$

Wave equation: $\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$
c = "velocity"

$y(x,t) \propto \psi(x) \begin{cases} \cos(\omega t) \\ \sin(\omega t) \end{cases}$ ω = (radial) frequency

particular solution of the wave equation

Find $\psi(x)$.

$y|_{x=0} = 0 \qquad y|_{x=L} = 0$

Equation for $\psi(x)$: $\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = -\frac{\omega^2}{c^2}\psi \begin{cases} \cos \\ \sin \end{cases}$$

$$\frac{\partial^2 y}{\partial x^2} = \psi'' \begin{cases} \cos \\ \sin \end{cases}$$

k^2: wavenumber^2

$$\psi''(x) + \frac{\omega^2}{c^2}\psi(x) = 0$$

2nd order ODE for $\psi(x)$

$$\psi = A\cos(kx) + B\sin(kx), \quad 0 \le x \le L$$

homogeneous boundary conditions

$\boxed{\begin{array}{l} \psi(x=0) = 0 \\ \psi(x=L) = 0 \end{array}}$

$\psi(x=0)=0 \rightarrow A = 0$; $\psi = B\sin(kx)$

$\psi(x=L)=0 \rightarrow B\sin(kL) = 0 \xrightarrow{B \neq 0} \sin(kL) = 0 \rightarrow kL = n\pi$, $n = 1, 2, \ldots$

$\psi \neq 0$ for some x

$$\boxed{k = \frac{n\pi}{L}} \rightarrow \boxed{\omega_n = \frac{n\pi c}{L}}$$

characteristic frequencies of the string