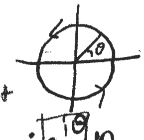
Ⓘⅈⅈ Integrals $I = \int_0^{2\pi} d\theta\, F(\sin\theta, \cos\theta)$

ex $\quad I = \int_0^{2\pi} \frac{d\theta}{A + B\sin\theta} \qquad A, B: \text{real}, \quad A > |B| \geq 0$

1. let $z = e^{i\theta}$

z moves around unit circle



$\int_0^{2\pi} \to \oint_c$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(e^{i\theta} - \frac{1}{e^{i\theta}}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = i\underbrace{\boxed{e^{i\theta}}}_{z}d\theta \to d\theta = \frac{dz}{iz}$$

$$\left[\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)\right]$$

2. $I = \oint_c \frac{\frac{dz}{iz}}{A + B\left(\frac{1}{2i}\left[z - \frac{1}{z}\right]\right)} = \frac{2iz}{B}\oint_c \frac{dz/iz}{\underset{\substack{\uparrow \\ \text{coefficient is 1}}}{z^2} + \frac{2Ai}{B}z - 1} = \frac{2}{B}\oint_c dz \frac{1}{z^2 + \frac{2iA}{B} - 1}$

find the singularities of the integrand. $z^2 + \frac{2iA}{B}z - 1$

$$z_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1}{\cancel{2}}\left(\frac{-2iA}{B} \pm \sqrt{\frac{-\cancel{4}A^2}{B^2} + \cancel{4}}\right) \qquad \text{simple zeros}$$

$$= \frac{-iA}{B} \pm \underbrace{\sqrt{\frac{-A^2}{B^2} + 1}}_{<0} = \frac{-iA}{B} \pm i\sqrt{\frac{A^2}{B^2} - 1}$$

$$= -i\left(\frac{A}{B} \mp \sqrt{\frac{A^2}{B^2} - 1}\right)$$

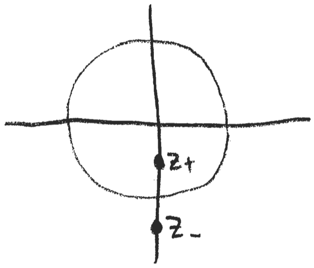
$z_+ = -i\left[\frac{A}{B} - \sqrt{\frac{A^2}{B^2} - 1}\right] \qquad z_- = -i\left[\frac{A}{B} + \sqrt{\frac{A^2}{B^2} - 1}\right]$

$= -i \cdot p \qquad\qquad = -i \cdot q \qquad\qquad (B > 0)$

$p = \frac{A}{B} - \sqrt{\frac{A^2}{B^2} - 1} \; ; \; p > 0$

$\boxed{p < 1}$?. ✓ $\frac{A}{B} - \sqrt{\frac{A^2}{B^2} - 1} < 1 \Leftrightarrow \frac{A}{B} - 1 < \sqrt{\frac{A^2}{B^2} - 1} \to \left(\frac{A}{|B|} + 1\right)\left(\frac{A}{|B|} + 1\right)$

$\sqrt{\frac{A}{|B|} - 1} < \sqrt{\frac{A}{|B|} + 1}$



$az^2 + bc + c = 0$ w/roots z_1, z_2

$z_1 \cdot z_2 = \frac{c}{a} = -1$

$z_+ \cdot z_1 = -1 \;, \; |z_+| = \frac{1}{|z_+|} > 1$

Residue Theorem:

$$J = 2\pi i \operatorname*{Res}_{z=z_+} \frac{2}{B} \frac{1}{z^2 + \frac{2iA}{B} z - 1}$$

$$= 2\pi i \cdot \frac{2}{B} \frac{1}{2z + 2iA/B} = \frac{2\pi}{\sqrt{A^2 - B^2}}$$