BEN HARRIS: Hi. Today we're going to do a problem about similar matrices. The problem is right here. So the question asks, which of the following statements are true, and it asks you to explain why.

The first statement is: if A and B are similar matrices, then 2 A cubed plus A minus 3 times the identity is similar to 2 times B cubed plus B minus 3 times the identity. The second question asks: if A and B are 3 by 3 matrices with eigenvalues 1, 0, and -1, then they're similar. And the third part asks you whether these two J matrices are similar.

OK. I'll give you a moment to hit pause and try it on your own. I'll be back in just a moment, and we can do it together.

And we're back. Let's start with part A. Part A is true. Why? Well, what does it mean for A and B to be similar? Right, we know there's some matrix M, such that-- that's what "st" means-- when I multiply A on the left by M and the right by M inverse, I get B.

OK. So let's take that same matrix M and multiply it on the left and the right of 2 A cubed plus A minus 3 times the identity.

OK. What do we get here? Well, the point is what M times A cubed times M inverse, we can just write that as three M A M inverses. Similarly, we have an M*A M inverse and an M times the identity times M inverse.

Good. Remember that M times A times M inverse is B. So we just get 2 B cubed-- three B's-- plus B minus-- well, M times the identity is just M. So we just get the identity back, and we have 3 times the identity.

Good. And this is a general remark, that if you have matrices A and B that are similar, then any polynomials in these matrices A and B will be similar. It's the exact same justification. OK.

Now let's go on to part B. So now A and B are 3 by 3 similar matrices with the same eigenvalues. And their eigenvalues are distinct. So it turns out that b is true as well. And why is that? A matrix with distinct eigenvalues is diagonalizable. So we can write A as S lambda S inverse, where lambda is just this eigenvalue matrix. We can also write B as T lambda T inverse, where lambda is the same in both cases because they have the same eigenvalues.

Good. Now I'll let you-- so before we check it, let's just say the point. The point is if two matrices are similar to the same matrix, then they're similar to each other. Similarity is a transitive relation. And I'll just let you check that you can take T S inverse A times T S inverse inverse, and you'll get B. This follows directly from these two relations.

Good. Now let's take on part C. Part C is false. Let's come back over here and look at these two matrices, J_1 and J_2. The first thing you should see is that these two are Jordan blocks-- sorry, not Jordan blocks, they're matrices in Jordan normal form. They're different matrices in Jordan normal form, so they will not be similar. But let's actually see why.

Let's look at-- remember, one of the things that similarity preserves are eigenvectors and eigenvalues. So let's look at the eigenspace with eigenvalue minus one with these two matrices. So J_1 plus the identity-- let's look at the nullspace of this matrix. So this is just 0's on the diagonal and 1's right above the diagonal. And J_2 plus the identity. this is just 0, 1, 0, 0.

So the point is that the nullspace of this matrix is just one-dimensional. So there's only one independent eigenvector of J_1 with eigenvalue minus 1. Whereas the nullspace of this matrix is two-dimensional. There are two independent eigenvectors with eigenvalue minus 1. So the dimension-- the nullspace of J_1 plus the identity, this is 1, and this is 2. So they cannot possibly be similar.

Good. So that completes the problem. It's a nice exercise to do this more generally. And you can use these techniques not just looking at the number of independent eigenvectors and the nullspace of your J minus lambda*I matrix, but also powers of J minus lambda I and their nullspaces. You can use this to show that any two matrices in Jordan normal form that are different are not similar. This same method works. And that's a nice exercise if you want to go a little further with similar matrices.

Let's just recap the properties we saw here. We saw that if we had two similar matrices, then any polynomials in those matrices were similar. And we saw that if we have two matrices that have the same distinct eigenvalues, then they're similar. And we saw that, in a special case, we saw that two matrices in Jordan normal form that are different are not similar. Thanks.