

Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra*: Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$
$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution: Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, the vectors \mathbf{v}_4 , \mathbf{v}_5 , and \mathbf{v}_6 are dependent on the vectors \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 . To determine the relationship between the vectors \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 we apply row reduction to the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are independent. Therefore the largest number of independent vectors among the given six vectors is **three**. This will be the rank of the 4 by 6 matrix of \mathbf{v} 's.

Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in \mathbb{R}^3 . Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

Solution: This plane is the nullspace of the matrix $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ and also

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The special solutions to $A\mathbf{x} = \mathbf{0}$ are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

These form a basis for the nullspace of A and thus for the plane.

The intersection of this plane with the xy plane contains \mathbf{v}_1 and does not contain \mathbf{v}_2 ; the intersection must be a line. Since \mathbf{v}_1 lies on this line it also provides a basis for it.

Finally, we can use “inspection” or the cross product to find the vector

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix},$$

which is perpendicular to both \mathbf{v}_1 and \mathbf{v}_2 . It is therefore perpendicular to the plane. Since the space of vectors perpendicular to a plane in \mathbb{R}^3 is one-dimensional, \mathbf{v}_3 serves as a basis for that space.

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