PROPERTIES OF SIMPLE ROOTS

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Let V be a Euclidean space, that is a finite dimensional real linear space with a symmetric positive definite inner product $\langle , \rangle$.

Recall that for a root system Δ in V, A subset $\Pi \subset \Delta$ is a set of simple roots (a simple root system) if

(1) Π is a basis in V;
(2) Each root $\beta \in \Delta$ can be written as a linear combination of elements of Π with integer coefficients of the same sign, i.e.
$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$$
with all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$.

The root β is positive if the coefficients are nonnegative, and negative otherwise. The set of all positive roots (positive root system) associated to Π is denoted Δ^+.

Below we will assume that the root system Δ is reduced, that is, for any $\alpha \in \Delta, 2\alpha \notin \Delta$.

Consider a given abstract root system Δ and some simple root system $\Pi \subset \Delta$. We may represent $\beta \in \Delta$ as a linear combination of elements of Π with integer coefficients of the same sign. By definition, we know that the elements of Π are linearly independent. Thus, such a representation of β is unique and exactly one of $\pm\beta$ is a positive root associated to Π. Δ^+ is simply the set of all positive roots associated to Π, and so our choice of simple root system determines an associated set of positive roots uniquely.

Now suppose instead that we are given a positive root system $\Delta^+ \subset \Delta$. We can construct a vector $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, and such that $\Delta^+ = \Delta_t^+$, where
$$\Delta_t^+ = \{\alpha \in \Delta : \langle t, \alpha \rangle > 0\}.$$
We can be sure from [3] that such a vector always exists, and by Definition 4 in [3], we can determine the set of all indecomposable elements in Δ_t^+. Lets call this set Π_t. By Theorem 9 in [3], $\Pi = \Pi_t$, and we have now proven our first theorem.

Theorem 1. *In a given Δ, a set of simple roots $\Pi \subset \Delta$ and the associated set of positive roots $\Delta^+ \subset \Delta$ determine each other uniquely.*

Date: 10th May 2004.

The question of existence of sets of simple roots for any abstract root system Δ is settled in [3]. Theorem 1 shows that once Π is chosen Δ^+ is unique. In this paper we want to address the question of the possible choices for $\Pi \subset \Delta$. We start with a couple of examples.

Example 2. *The root system Δ of the type A_2 consists of the six vectors $\{e_i - e_j\}_{i \neq j}$ in V, the plane orthogonal to the line $e_1 + e_2 + e_3$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$.*

We can easily present the vectors of this root system in a standard orthonormal basis of the plane. Let $i = (e_r - e_t)/\sqrt{2}$ and $j = (2e_s - e_r - e_t)/\sqrt{6}$ such that $r, s, t \in \{1, 2, 3\}$ are distinct.

$$\sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2} = \|i\| = 1 = \|j\| = \sqrt{(\frac{-1}{\sqrt{6}})^2 + (\frac{2}{\sqrt{6}})^2 + (\frac{-1}{\sqrt{6}})^2}$$

$$\langle i, j \rangle = \sqrt{(\frac{1}{\sqrt{2}})(\frac{-1}{\sqrt{6}}) + (0)(\frac{2}{\sqrt{6}}) + (\frac{-1}{\sqrt{2}})(\frac{-1}{\sqrt{6}})} = 0 \ .$$

So $\{i, j\}$ is an orthonormal basis in V and $\Delta = \{\pm\sqrt{2}i, \pm\frac{\sqrt{2}}{2}i \pm \frac{\sqrt{6}}{2}j\}$.

By Lemmas 6 and 7 in [3] we know that any simple root system must be linearly independent with an obtuse angle between each pair of simple roots. Every subset in Δ that satisfies these conditions is a pair of vectors that form a $120°$ angle. We know that the sum of two vectors of length l separated by $120°$ is a third vector of length l that bisects the original two:

$$\begin{pmatrix} l\cos(60°) \\ l\sin(60°) \end{pmatrix} + \begin{pmatrix} l\cos(-60°) \\ l\sin(-60°) \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix}.$$

Therefore the possibilities for the positive roots system Δ^+ are any set of three adjacent vectors in Δ; for any Δ^+, its associated simple root system $\Pi \subseteq \Delta^+$ is the set of two outer vectors. In [1] we learned that rotations are a type of orthogonal operator. Then any two simple root systems, and their associated set of positive roots, can be mapped to each other by an orthogonal transformation, namely a rotation of $60k$ degrees, $k \in \mathbb{Z}$:

$$\begin{pmatrix} \cos(60k°) & -\sin(60k°) \\ \sin(60k°) & \cos(60k°) \end{pmatrix} \begin{pmatrix} \sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \end{pmatrix} = \begin{pmatrix} \sqrt{2}\cos(\theta + 60k°) \\ \sqrt{2}\sin(\theta + 60k°) \end{pmatrix}.$$

Example 3. *Consider the root system Δ of the type B_2 in $V = \mathbb{R}^2$: it consists of eight vectors $\{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}$.*

Just as in 2, any simple root system in B_2 must be linearly independent with an obtuse angle between each pair of simple roots. Every subset in Δ that satisfies these conditions is a pair of vectors that form a $135°$ angle. Consider one such subset $\Pi \subset \Delta$ and the two roots $\alpha, \beta \in \Pi$, $\|\alpha\| = 1$, $\|\beta\| = \sqrt{2}$. We can easily choose an orthonormal basis $\{i, j\} \subset V$ such that $i = \alpha\ ,\ \ -i + j = \beta$. It follows that $\beta + \alpha$ is the root, of length 1, $45°$ from β and $90°$ from α, and $\beta + 2\alpha$ is the root, of length $\sqrt{2}$, $90°$ from β and

$45°$ from α. Then $\Pi = \{\alpha, \beta\}$ is a simple root system with the associated positive root system $\Delta^+ = \{\beta, \beta+\alpha, \beta+2\alpha, \alpha\}$. To generalize in B_2, a positive root system Δ^+ is a set of four adjacent roots, which are associated to the set of two outer roots, the associated simple root system Π. It is easy to see that, as in 2, a rotation is a type of orthogonal transformation that maps $\Delta^+, \Pi \subseteq \Delta^+$ to other positive and simple root systems. In this case, rotations of $90k$ degrees, $k \in \mathbb{Z}$ will yield 4 distinct simple and positive root systems. The 4 reflections $s_\gamma, \gamma \in \Delta^+$ will return the other 4 possible simple and positive root systems. This is one example of how a reflection operating on Δ^+ returns a positive root system distinct from those achieved through rotation:

$$s_\alpha(\{\beta, \alpha\}) = \{i+j, -i\}\ , \quad s_\alpha(\{\beta, \beta+\alpha, \beta+2\alpha, \alpha\}) = \{i+j, j, -i+j, -i\}.$$

The remaining three can be just as easily verified, especially with a well-chosen orthonormal basis. Like rotations, reflections are a type of orthogonal transformation (from [1]).

Let us start working towards a result generalizing our observations. Recall the definition of a reflection associated to an element $\alpha \in V$ (from [1]):

$$s_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

It is an orthogonal transformation of V. Let $\Pi \subset \Delta$ be a simple root system associated to $\Delta^+ \subset \Delta$. Suppose there are m simple roots in Π. Consider the m inequalities $\langle t, \alpha\rangle > 0$ for all $\alpha \in \Pi$, where t is in the span of Π. t is a variable m-dimensional vector, and all $\alpha \in \Pi$ are known. We have m inequalities in m variables, a solvable linear system since Π is a basis in $\mathbb{R}^m$, and t exists. Theorem 9 in [3] tells us that $\Pi = \Pi_t\ ,\ \ \Delta^+ = \Delta_t^+$. For any $\alpha \in \Delta$ the reflection s_α maps the set $\{\beta_i\}_{i=1}^n$ of n roots in Δ_t^+ to the set $\{s_\alpha(\beta_i)\}_{i=1}^n$ of n roots in Δ. From [1] we know that orthogonal transformations preserve the inner product of two vectors. This tells us that

$$\langle s_\alpha(\beta_i), s_\alpha(t)\rangle = \langle \beta_i, t\rangle\ ,\ i = 1, 2, \ldots, n.$$

Once again Theorem 9 in [3] helps us recognize that $s_\alpha(\Delta_t^+) = \Delta_{s_\alpha(t)}^+$. Furthermore,

$$\delta = \beta + \gamma \Longrightarrow s_\alpha(\delta) = s_\alpha(\beta) + s_\alpha(\gamma);$$
$$s_\alpha(s_\alpha(\delta)) = s_\alpha(s_\alpha(\beta)) + s_\alpha(s_\alpha(\gamma)) = \beta + \gamma = \delta.$$

Hence δ is indecomposable if and only if $s_\alpha(\delta)$ is indecomposable with respect to $s_\alpha(\Delta^+)$, and

$$s_\alpha(\Pi_t) = \Pi_{s_\alpha(t)} \subseteq \Delta_{s_\alpha(t)}^+ = s_\alpha(\Delta_t^+).$$

We have now proven the following theorem.

Theorem 4. *Let $\Pi \subset \Delta$ be a set of simple roots, associated to the set of positive roots Δ^+. For any $\alpha \in \Delta$, the set obtained by reflection $s_\alpha(\Pi)$ is a simple root system with the associated positive root system $s_\alpha(\Delta^+)$.*

To understand better the passage from Δ^+ to $s_\alpha(\Delta^+)$, we consider the special case when α is a simple root. Then $\Delta^+ \cup s_\alpha(\Delta^+)$ differs from $\Delta^+ \cap s_\alpha(\Delta^+)$ by exactly two roots: $\pm\alpha$.

Theorem 5. *Let $\Pi \subset \Delta$ be a simple root system, contained in a positive root set Δ^+. If $\alpha \in \Pi$, then the reflection s_α maps the set $\Delta^+ \setminus \{\alpha\}$ to itself.*

Let $\beta \in \Delta^+ \setminus \{\alpha\}$ where $\alpha \in \Pi$. Then β can be represented as a linear combination of simple roots with nonnegative integer coefficients:

$$\beta = c_\alpha \alpha + \sum_{\gamma \in \Pi \setminus \{\alpha\}} c_\gamma \gamma \Longrightarrow s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = \sum_{\gamma \in \Pi \setminus \{\alpha\}} c_\gamma \gamma + (c_\alpha - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle})\alpha.$$

Since all $c_\gamma \geq 0$, not all zero, then the coefficient $c_\alpha - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ must be non-negative, and $s_\alpha(\beta) \in \Delta^+ \setminus \{\alpha\}$. By Theorem 15 in [1] we know that s_α maps the n distinct roots in Δ to n distinct roots in Δ. Therefore s_α maps the m distinct roots in $\Delta^+ \setminus \{\alpha\}$ to m distinct roots in $\Delta^+ \setminus \{\alpha\}$. In other words, $s_\alpha(\Delta^+ \setminus \{\alpha\}) = \Delta^+ \setminus \{\alpha\}$, as was to be proved.

Corollary 6. *Any two positive root systems in Δ can be obtained from each other by a composition of reflections with respect to the roots in Δ.*

The proof of this corollary follows easily from 5. Let n be the number of roots in $\Delta_1^+ \cap \Delta_2^-$ such that $\Delta_1^+, \Delta_2^+ \subset \Delta$. We can prove the claim by induction on n. Suppose $n = 0$. Then $\Delta_1^+ = \Delta_2^+$ and so the composition $s_\alpha(s_\alpha(x))$ will suffice, where α may be any vector in the space, including any root of Δ. So we know the claim is true for $n = 0$. Now suppose $n \geq 1$. Let Π_1 be the simple root system associated to Δ_1^+. Then there exists a root $\alpha \in \Pi_1 \subseteq \Delta_1^+$ such that $\alpha \in \Delta_2^-$. If this were not the case, then $\Pi_1 \cap \Delta_2^-$ would be empty, which implies $\Pi_1 \subseteq \Delta_2^+$. But every root in Δ_1^+ is a positive sum of the simple roots in Π_1, which themselves are a positive sum of the simple roots in the simple root system Π_2 associated to Δ_2^+. Then the m positive roots in Δ_1^+ would be the same as the m positive roots in Δ_2^+ such that $\Delta_1^+ \cap \Delta_2^-$ is empty, which would contradict $n \geq 1$. So there are $n-1$ roots in $\Delta_1^+ \setminus \{\alpha\} \cap \Delta_2^- \setminus \{\alpha\}$. By 5 we know that the reflection s_α will map $\Delta_1^+ \setminus \{\alpha\} \to \Delta_1^+ \setminus \{\alpha\}$ and we also know that $s_\alpha(\alpha) = -\alpha$. Since not both $\pm\alpha$ can be in Δ_2^-, $s_\alpha(\Delta_1^+) \cap \Delta_2^-$ has $n-1$ roots and we see how the reflection s_α decreased n by one. If the claim is true for $n-1$, then the claim must be true for n. So the claim is true for all $n \geq 0$.

The statements above show that although a set of simple roots is not unique for a given Δ, they are related to each other by a simple orthogonal transformation of the space V. In particular, the angles and relative lengths of simple roots in any two simple root systems in Δ are the same. The next theorem proves another useful property of simple roots.

Theorem 7. *Let Δ^+ be a positive root system in Δ such that $\Pi \subseteq \Delta^+$ is its associated simple root system. Any positive root $\beta \in \Delta^+$ can be written* *as a sum*

$$\beta = \alpha_1 + \alpha_2 + \ldots + \alpha_k,$$

where $\alpha_i \in \Pi$ for all $i = 1, \ldots, k$ (repetitions are allowed). Moreover, it can be done so that each partial sum

$$\alpha_1 + \ldots + \alpha_m, \ \ 1 \leq m \leq k$$

is also a root.

The claim can be proved by induction. Choose a vector t that is in the span of Π such that $\langle t, \alpha \rangle = 1$ for all $\alpha \in \Pi$. We know such a t exists by a similar argument used in the proof of 4. For any given positive root $\beta \in \Delta^+$ we can choose a simple root in Π such that its inner product with β is positive, and call it α_r, where r is the value of $\langle t, \beta \rangle$. We know such a simple root exists because otherwise Lemma 7 in [3] tells us that $\Pi \cup \{\beta\}$ would be linearly independent, which it clearly is not. Since all $\beta \in \Delta^+$ are equal to a linear combination of simple roots with nonnegative coefficients, it must be true that $\langle t, \beta \rangle$ is the sum of those coefficients and $r \geq 1$.

By induction on r we can prove the claim is true for all positive roots $\beta \in \Delta^+$. Suppose $r = 1$. Then $\beta = \alpha_r$ is a simple root and the claim is obviously true. Now suppose $r \geq 2$. By Theorem 10 in [2], $\beta - \alpha_r$ is a root or 0. It must be a root since $0 \neq \langle t, \beta - \alpha_r \rangle = r - 1 \geq 1$. We also know that

$$\beta = c_{\alpha_r}\alpha_r + \sum_{\alpha \in \Pi \setminus \{\alpha_r\}} c_\alpha \alpha \ , \ \ \beta - \alpha_r = (c_{\alpha_r} - 1)\alpha_r + \sum_{\alpha \in \Pi \setminus \{\alpha_r\}} c_\alpha \alpha$$

where all c_α are nonnegative and not all zero. All c_α are nonnegative and not all zero so both $c_{\alpha_r} - 1$ and all $c_{\alpha \neq \alpha_r}$ must be nonnegative and not all zero. This means that $\beta - \alpha_r$ is a positive root. Because $\beta = (\beta - \alpha_r) + \alpha_r$ where $\alpha_r \in \Pi$, if the claim is true for $r - 1$ then it must be true for r. And since the claim is true for $r = 1$, then it must be true for all $r \geq 1$, as was to be proved.

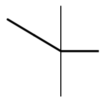
We have now developed the tools to move on to some interesting examples.

Example 8. *Let Δ be the root system in $V = \mathbb{R}^2$ such that the angle between the simple roots is $\frac{5\pi}{6}$.*

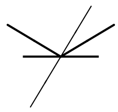
This condition determines Δ completely (this is the root system of the type G_2). Let $\Pi = \{\alpha, \beta\}$ be a simple root system in Δ.

$$\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \cdot \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = 4\cos^2\theta = 3$$

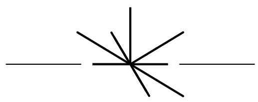
Note that both multiplicands on the left hand side are integers by Definition 2 in [2] and nonpositive by Lemma 6 in [3]. Since their product is 3 we know that one must be equal to -3 (assume the left one) and the other must be equal to -1 (assume the right one). Then $\frac{\|\beta\|}{\|\alpha\|} = \sqrt{3}$. The sketch below shows Π (all sketches not drawn exactly to scale).



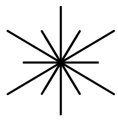
Notice that the 2 thick lines represent the simple root system Π while the thin vertical line represents the hyperplane of α. Since $s_\gamma(\Delta) = \Delta$, for all $\gamma \in \Delta$, then the set $\Pi \cup s_\alpha(\Pi)$ must be in Δ, and it is sketched below.



This time the thin line represents the hyperplane of β. Just as before, we can expand our set of roots so this time it looks like the sketch below.



In this diagram the disjoint thin line represents the hyperplane of the root perpendicular to α. A third and final reflection returns the full abstract root system G_2, shown below.



We know from [2] that the minimum angle between roots is $30°$, so no other roots are possible. This is due to the limitations put onto the angle between roots and their relative lengths outlined in Definition 2-(3) from [2] and Theorem 8-(3) from [2]. We may apply Theorem 7 in this case to present each positive root as a sum of simple roots. First, define an orthonormal basis $\{i, j\} \subset \mathbb{R}^2$ such that $\alpha = i$ and $\beta = -\frac{3}{2}i + \frac{\sqrt{3}}{2}j$. We may use elementary geometry to conclude

$$\{\alpha, \beta\} \ = \Pi \subseteq \Delta^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha, 2\beta + 3\alpha, \alpha\}.$$

In this way, we see that any positive root in G_2 can be presented as the partial sum described in 7.

Recall that two root systems Δ and Δ' are isomorphic if there exists an orthogonal transformation of V that maps Δ to Δ'. A root system is irreducible if it cannot be decomposed as a disjoint union of two root systems $\Delta = \Delta' \cup \Delta''$ of smaller dimension, so that each element of Δ' is orthogonal to each element of Δ''.

Example 9. *There are just three non-isomorphic irreducible root systems in $V = \mathbb{R}^3$, of the types A_3, B_3 and C_3 (see Example 5 in* [2]*, Examples 10 and 11 in* [3] *for definitions).*

Any other abstract root system $\Delta \subset V$ must be a reducible system of the type $\Delta = \Delta' \cup \Delta''$, where Δ', Δ'' are two root systems of smaller dimension such that all roots in Δ' are orthogonal to all roots in Δ''. Clearly, one of Δ', Δ'' is 1-dimensional (say Δ') and the other is 2-dimensional (say Δ''). The only 1-dimensional abstract root system is A_1, so Δ' must be of this type. As for Δ'', the only 2-dimensional abstract root systems are $A_1 \oplus A_1, A_2, B_2$, and G_2. A definition of an abstract root system can be found in [2]. Using this definition, we can verify that there are four reducible root systems $\Delta_i \subset \mathbb{R}^3$, $i = A_1 \oplus A_1, A_2, B_2, G_2$, with $\Delta_i = \Delta_i' \cup \Delta_i''$. In each of the four cases, let Δ_i' be a root system of type A_1 and let Δ_i'' be a root system of type i such that every root in Δ_i'' is orthogonal to every root in Δ_i'. The first criterion int he definition is met in all four cases because Δ_i' and Δ_i'' are linearly independent with dimensions 1 and 2, respectively. Therefore their union spans $\mathbb{R}^3$. If α is a root in Δ_i, then α is in exactly one of Δ_i' or Δ_i''. If $\alpha \in \Delta_i'$, then the reflection s_α will map $\Delta_i' \to \Delta_i'$, since Δ_i' is an abstract root system, and will leave Δ_i'' unchanged by orthogonality. The same applies for $\alpha \in \Delta_i''$. This shows that the second criterion is satisfied. The third criterion refers to any two roots in Δ_i. If the two roots are from the same abstract root system, then we already know the value is an integer. If one root is from Δ_i' and the other is from Δ_i'', then the value must be 0 by orthogonality. Either way, the value is an integer and the third and final criterion is satisfied. Therefore, all of the four possibilities for Δ_i are abstract root systems.

References

[1] Bergren, Anna, *Reflections in a Euclidean space*, preprint, MIT, 2004.
[2] Herman, Matthew, *Abstract root systems*, preprint, MIT, 2004.
[3] Valkama, Juha, *Simple and positive roots*, preprint, MIT, 2004.