ABSTRACT ROOT SYSTEMS

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Let V be a Euclidean space, that is, a real, finite-dimensional vector space with a symmetric, positive-definite inner product $\langle\ ,\ \rangle$. Recall the definition of a reflection in V from [1]:

Definition 1. A **reflection** of a vector $\vec{x} \in V$ with respect to a vector $\vec{\alpha} \in V$ is defined by the formula

$$s_{\vec{\alpha}}(\vec{x}) = \vec{x} - \frac{2\langle \vec{x}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \vec{\alpha}\,.$$

We can now define an abstract root system in a Euclidean space.

Definition 2. An **abstract root system** in V is a finite set Δ of nonzero elements of V such that

(1) Δ spans V;
(2) for all $\vec{\alpha} \in \Delta$, the reflections

$$s_{\vec{\alpha}}(\vec{\beta}) = \vec{\beta} - \frac{2\langle \vec{\beta}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \vec{\alpha}$$

map the set Δ to itself;
(3) the number $\dfrac{2\langle \vec{\beta}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle}$ is an integer for any $\vec{\alpha}, \vec{\beta} \in \Delta$.

A **root** is an element of Δ.

We will begin by considering some examples of root systems.

Example 3. Let V be the following subspace of $\mathbb{R}^{n+1}$, $n \geq 1$:

$$V = \left\{ \sum_{i=1}^{n+1} a_i \vec{e}_i, \text{ with } \sum_{i=1}^{n+1} a_i = 0 \right\}, \tag{1}$$

where $\{\vec{e}_i\}_{i=1}^{n+1}$ is an orthonormal basis in $\mathbb{R}^{n+1}$, and all $a_i \in \mathbb{R}$.

Claim. *The set $\Delta = \{\vec{e}_i - \vec{e}_j, i \neq j\}$ is an abstract root system.*

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Proof. We must first show that Δ spans V. Construct $\widetilde{\Delta} \subset \Delta$ where

$$\widetilde{\Delta} = \{\vec{e}_2 - \vec{e}_1, \vec{e}_3 - \vec{e}_1, \vec{e}_4 - \vec{e}_1, \ldots, \vec{e}_n - \vec{e}_1, \vec{e}_{n+1} - \vec{e}_1\}\,.$$

If we show that $\widetilde{\Delta}$ spans V, then Δ necessarily spans V as well.

A vector $\vec{v} \in V$ can be written as

$$\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n + a_{n+1}\vec{e}_{n+1} \tag{2}$$

where

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = 0\,. \tag{3}$$

Rewrite (3) as

$$a_1 = -(a_2 + a_3 + \cdots + a_n + a_{n+1}) \tag{4}$$

and substitute (4) into (2) to get

$$\vec{v} = -(a_2 + a_3 + \cdots + a_n + a_{n+1})\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n + a_{n+1}\vec{e}_{n+1}\,. \tag{5}$$

We can then simplify (5):

$$\vec{v} = a_2(\vec{e}_2 - \vec{e}_1) + a_3(\vec{e}_3 - \vec{e}_1) + \cdots + a_n(\vec{e}_n - \vec{e}_1) + a_{n+1}(\vec{e}_{n+1} - \vec{e}_1)\,. \tag{6}$$

Equation (6) clearly shows that any $\vec{v} \in V$ can be written as a linear combination of the elements of $\widetilde{\Delta}$. Hence, $\widetilde{\Delta}$ spans V, and therefore Δ spans V.

Next, we must show that for any $\vec{\alpha}, \vec{\beta} \in \Delta$, the reflections $s_{\vec{\alpha}}(\vec{\beta})$ map the set Δ to itself. Take $\vec{\alpha} = \vec{e}_i - \vec{e}_j$ and $\vec{\beta} = \vec{e}_k - \vec{e}_m$, where $i \neq j$ and $k \neq m$. Apply the reflection:

$$s_{\vec{e}_i - \vec{e}_j}(\vec{e}_k - \vec{e}_m) = \vec{e}_k - \vec{e}_m - \frac{2\langle \vec{e}_k - \vec{e}_m, \vec{e}_i - \vec{e}_j\rangle}{\langle \vec{e}_i - \vec{e}_j, \vec{e}_i - \vec{e}_j\rangle}(\vec{e}_i - \vec{e}_j) \tag{7}$$

By the symmetry and bilinearity of the inner product, we can simplify (7) as follows:

$$s_{\vec{e}_i - \vec{e}_j}(\vec{e}_k - \vec{e}_m) = \vec{e}_k - \vec{e}_m - \frac{2[\langle \vec{e}_k, \vec{e}_i\rangle - \langle \vec{e}_k, \vec{e}_j\rangle - \langle \vec{e}_m, \vec{e}_i\rangle + \langle \vec{e}_m, \vec{e}_j\rangle]}{\langle \vec{e}_i, \vec{e}_i\rangle - \langle \vec{e}_i, \vec{e}_j\rangle - \langle \vec{e}_j, \vec{e}_i\rangle + \langle \vec{e}_j, \vec{e}_j\rangle}(\vec{e}_i - \vec{e}_j)\,. \tag{8}$$

Since $\{\vec{e}_i\}_{i=1}^{n+1}$ is an orthonormal basis, we know that $\langle \vec{e}_i, \vec{e}_i\rangle = 1$ and $\langle \vec{e}_i, \vec{e}_j\rangle = 0$ if $i \neq j$. We can simplify the fraction in (8). The denominator is clearly 2, which cancels the 2 in the numerator. Hence,

$$\begin{aligned} s_{\vec{e}_i - \vec{e}_j}(\vec{e}_k - \vec{e}_m) &= \vec{e}_k - \vec{e}_m - [\langle \vec{e}_k, \vec{e}_i\rangle \\ &\quad - \langle \vec{e}_k, \vec{e}_j\rangle - \langle \vec{e}_m, \vec{e}_i\rangle + \langle \vec{e}_m, \vec{e}_j\rangle]\,(\vec{e}_i - \vec{e}_j)\,. \end{aligned} \tag{9}$$

It therefore follows that

$$s_{\vec{e}_i - \vec{e}_j}(\vec{e}_k - \vec{e}_m) = \begin{cases} \vec{e}_m - \vec{e}_k & \text{if } i = k, j = m \text{ or } i = m, j = k \\ \vec{e}_k - \vec{e}_m & \text{if } i \neq k \neq j \neq m \neq i \\ \vec{e}_j - \vec{e}_m & \text{if } i = k, j \neq m \\ \vec{e}_k - \vec{e}_j & \text{if } i = m, j \neq k \\ \vec{e}_i - \vec{e}_m & \text{if } i \neq m, j = k \\ \vec{e}_k - \vec{e}_i & \text{if } j = m, i \neq k \end{cases}\,. \tag{10}$$

All possible cases in (10) are the difference of two distinct elements of the orthonormal basis, which is exactly the definition of elements of Δ, so Δ is invariant under reflection and the second property is satisfied. In addition, for all cases the denominator of the fraction in (8) is 2, exactly canceling the 2 in the numerator. The sum of the inner products in the numerator is a combination of 0s and 1s, so it is always an integer. Thus, the fraction

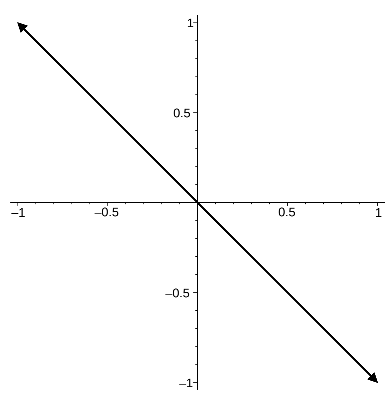
$$\frac{2\langle\vec{\alpha},\vec{\beta}\rangle}{\langle\vec{\alpha},\vec{\alpha}\rangle} = \frac{2[\langle\vec{e}_k,\vec{e}_i\rangle - \langle\vec{e}_k,\vec{e}_j\rangle - \langle\vec{e}_m,\vec{e}_i\rangle + \langle\vec{e}_m,\vec{e}_j\rangle]}{\langle\vec{e}_i,\vec{e}_i\rangle - \langle\vec{e}_i,\vec{e}_j\rangle - \langle\vec{e}_j,\vec{e}_i\rangle + \langle\vec{e}_j,\vec{e}_j\rangle}$$

is always an integer, which satisfies the third condition. □

Root systems as defined in example 3 are of the type A_n. We will now consider the geometry of A_1 and A_2.

For $n = 1$, V is the subspace of $\mathbb{R}^2$ where $V = \{a_1(\vec{e}_1 - \vec{e}_2) \,|\, a_1 \in \mathbb{R}\}$. Thus, $\Delta = \{\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_1\}$. The standard basis is orthonormal, so we can write $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$. As a result, $\Delta = \{(1, -1), (-1, 1)\}$ as depicted in figure 1.

FIGURE 1. A_1 Abstract Root System



If $n = 2$, $V = \{a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 \,|\, a_1 + a_2 + a_3 = 0\}$ is a subspace of $\mathbb{R}^3$. It is clear that $\Delta = \{\vec{e}_1 - \vec{e}_2, \vec{e}_1 - \vec{e}_3, \vec{e}_2 - \vec{e}_3, \vec{e}_2 - \vec{e}_1, \vec{e}_3 - \vec{e}_1, \vec{e}_3 - \vec{e}_2\}$. Using the standard basis in three-space, $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, and $\vec{e}_3 = (0, 0, 1)$, we observe that

$$\Delta = \{(1, -1, 0), (1, 0, -1), (0, 1, -1), (-1, 1, 0), (-1, 0, 1), (0, -1, 1)\},$$

which is shown geometrically in figure 2. If we connect the roots of A_2, we see that we get a regular hexagon of side length $\sqrt{2}$, as shown in figure 3.

Use of the standard basis is the simplest way to visualize A_1 and A_2, yet any orthonormal basis could have been chosen in each case.

FIGURE 2. A_2 Abstract Root System

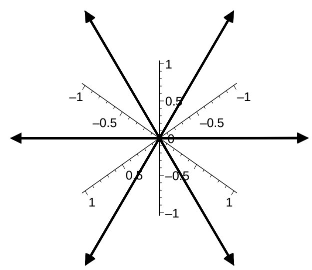
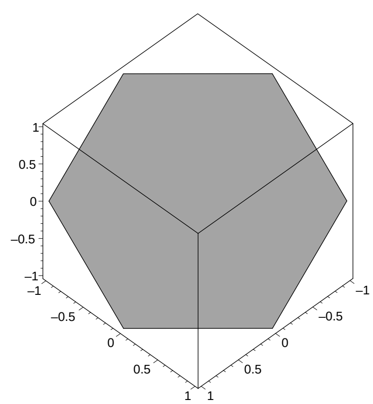


FIGURE 3. A_2 Abstract Root System (Viewed as a Polygon)



Example 4. Let V be the space $\mathbb{R}^n$ such that $n \geq 2$ with an orthonormal basis $\{\vec{e}_i\}_{i=1}^n$.

Claim. *The set* $\Delta = \{\pm\vec{e}_i \pm \vec{e}_j, i \neq j\} \cup \{\pm\vec{e}_i\}$ *is an abstract root system.*

Proof. The set Δ clearly spans V since it contains the orthonormal basis $\{\vec{e}_i\}_{i=1}^n$. We must next show that for all $\vec{\alpha}, \vec{\beta} \in \Delta$, the reflection $s_{\vec{\alpha}}(\vec{\beta}) \in \Delta$. Define $\{\vec{b}_i\}_{i=1}^n = \{\vec{e}_i\}_{i=1}^n$ where all $\vec{e}_i = \vec{b}_i$ and let $\vec{b}_{n+1} = \vec{0}$. We can now write Δ as $\Delta = \{\pm\vec{b}_i \pm \vec{b}_j, i \neq j\}$.

Take $\vec{\alpha} = \pm\vec{b}_i \pm \vec{b}_j$ and $\vec{\beta} = \pm\vec{b}_k \pm \vec{b}_m$, such that $i \neq j$ and $k \neq m$. We must show that $s_{\vec{\alpha}}(\vec{\beta}) \in \Delta$ for all $\vec{\alpha}, \vec{\beta} \in \Delta$. Begin by expanding the equation for the reflection:

$$s_{\pm\vec{b}_i \pm \vec{b}_j}(\pm\vec{b}_k \pm \vec{b}_m) = \pm\vec{b}_k \pm \vec{b}_m - \frac{2\langle \pm\vec{b}_k \pm \vec{b}_m, \pm\vec{b}_i \pm \vec{b}_j \rangle}{\langle \pm\vec{b}_i \pm \vec{b}_j, \pm\vec{b}_i \pm \vec{b}_j \rangle}(\pm\vec{b}_i \pm \vec{b}_j)\,. \tag{11}$$

We will first consider the fraction in (11). By the bilinearity of the inner product and by the orthonormality of the $\vec{b}_i$, we can simplify the denominator as follows:

$$\begin{aligned}\langle \pm\vec{b}_i \pm \vec{b}_j, \pm\vec{b}_i \pm \vec{b}_j\rangle &= \langle \pm\vec{b}_i, \pm\vec{b}_i\rangle + \langle \pm\vec{b}_i, \pm\vec{b}_j\rangle + \\ &\quad \langle \pm\vec{b}_j, \pm\vec{b}_i\rangle + \langle \pm\vec{b}_j, \pm\vec{b}_j\rangle \\ &= 1 + 0 + 0 + 1 \\ &= 2\,.\end{aligned}$$

This 2 in the denominator cancels the 2 in the numerator. We are now left with

$$s_{\pm\vec{b}_i \pm \vec{b}_j}(\pm\vec{b}_k \pm \vec{b}_m) = \pm\vec{b}_k \pm \vec{b}_m - \langle \pm\vec{b}_k \pm \vec{b}_m, \pm\vec{b}_i \pm \vec{b}_j\rangle(\pm\vec{b}_i \pm \vec{b}_j)\,. \tag{12}$$

Finally, simplify the remaining inner product.

$$\begin{aligned}\langle \pm\vec{b}_k \pm \vec{b}_m, \pm\vec{b}_i \pm \vec{b}_j\rangle &= \langle \pm\vec{b}_k, \pm\vec{b}_i\rangle + \langle \pm\vec{b}_k, \pm\vec{b}_j\rangle + \\ &\quad \langle \pm\vec{b}_m, \pm\vec{b}_i\rangle + \langle \pm\vec{b}_m, \pm\vec{b}_j\rangle\end{aligned} \tag{13}$$

It therefore follows that

$$s_{\pm\vec{b}_i \pm \vec{b}_j}(\pm\vec{b}_k \pm \vec{b}_m) = \begin{cases} \mp\vec{b}_i \mp \vec{b}_j & \text{if } i = k, j = m \text{ or } i = m, j = k \\ \pm\vec{b}_k \pm \vec{b}_m & \text{if } i \neq k \neq j \neq m \neq i \\ \pm\vec{b}_m \mp \vec{b}_j & \text{if } i = k, j \neq m \\ \pm\vec{b}_k \mp \vec{b}_j & \text{if } i = m, j \neq k \\ \pm\vec{b}_m \mp \vec{b}_i & \text{if } i \neq m, j = k \\ \pm\vec{b}_k \mp \vec{b}_i & \text{if } j = m, i \neq k \end{cases} \tag{14}$$

All 6 cases in (14) are elements of Δ, so the second property of an abstract root system is satisfied. Furthermore, all the possible cases in (14) indicate that the inner product in (13) can only be 0, ± 1, or ± 2. Hence, the fraction

$$\frac{2\langle \vec{\alpha}, \vec{\beta}\rangle}{\langle \vec{\alpha}, \vec{\alpha}\rangle} = \frac{2\langle \pm\vec{b}_k \pm \vec{b}_m, \pm\vec{b}_i \pm \vec{b}_j\rangle}{\langle \pm\vec{b}_i \pm \vec{b}_j, \pm\vec{b}_i \pm \vec{b}_j\rangle}$$

must always be an integer. As a result, the set $\Delta = \{\pm\vec{e}_i \pm \vec{e}_j, i \neq j\} \cup \{\pm\vec{e}_i\}$ is an abstract root system. □

We call this type of abstract root system B_n. Suppose $n = 2$. In this case, $V = \mathbb{R}^2$ and

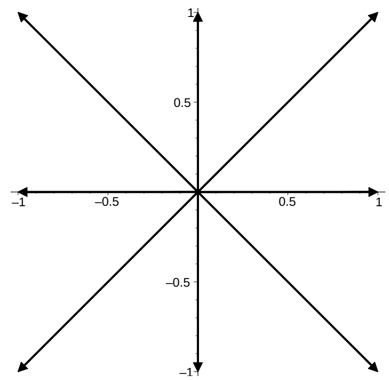
$$\Delta = \{\vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2, -\vec{e}_1 + \vec{e}_2, -\vec{e}_1 - \vec{e}_2, \vec{e}_1, -\vec{e}_1, \vec{e}_2, -\vec{e}_2\}\,.$$

We will once again choose the standard basis for $\mathbb{R}^2$ where $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$. Hence,

$$\Delta = \{(1,1), (1,-1), (-1,1), (-1,-1), (1,0), (-1,0), (0,1), (0,-1)\}\,,$$

which is depicted graphically in figure 4.

FIGURE 4. B_2 Abstract Root System



We will now consider the sizes and further classifications of abstract root systems. We begin with a simple definition.

Definition 5. An abstract root system is **reducible** if it can be represented as a disjoint union of two abstract root systems $\Delta = \Delta' \cup \Delta''$, and each element of Δ' is orthogonal to each element of Δ''. We say that Δ is **irreducible** if it admits no such decomposition.

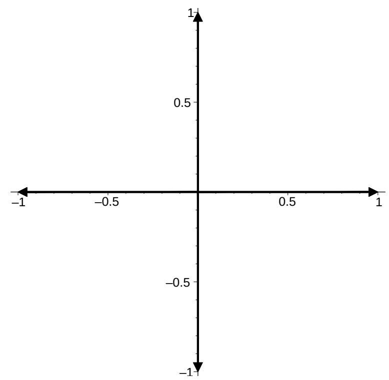
This definition motivates us to test our familiar root systems A_2 and B_2 to determine whether they are reducible. It is clear from figure 2 that no root system Δ_{A_2} of type A_2 is reducible since no two vectors in any Δ_{A_2} are orthogonal, and so there cannot possibly be two smaller root systems $\Delta'_{A_2}, \Delta''_{A_2} \subset \Delta_{A_2}$ where each element in Δ'_{A_2} orthogonal to each element in Δ''_{A_2}.

Now consider root systems of type B_2. From figure 4, it is equally obvious that there exists no reducible root system Δ_{B_2} of type B_2. In this case, each of the eight vectors in Δ_{B_2} is orthogonal to only one of the other vectors in in Δ_{B_2}. Hence, we cannot find two sets Δ'_{B_2} and Δ''_{B_2} that are orthogonal to each other.

An example of a reducible root system in $\mathbb{R}^2$ is $A_1 \oplus A_1$, which is the union of two A_1 root systems. Suppose $\Delta_{A_1 \oplus A_1}$ is an abstract root system of type $A_1 \oplus A_1$. That is, $\Delta_{A_1 \oplus A_1} = \{\vec{e}_1, -\vec{e}_1, \vec{e}_2, -\vec{e}_2\}$. Let $\Delta'_{A_1 \oplus A_1} = \{\vec{e}_1, -\vec{e}_1\}$ and $\Delta''_{A_1 \oplus A_1} = \{\vec{e}_2, -\vec{e}_2\}$. It is obvious that these two sets are orthogonal.

To further classify abstract root systems, we will prove some elementary theorems about them.

FIGURE 5. $A_1 \oplus A_1$ Abstract Root System



Theorem 6. *Let Δ be an abstract root system in V.*

(1) *If $\vec{\alpha} \in \Delta$, then $-\vec{\alpha} \in \Delta$.*
(2) *If $\vec{\alpha} \in \Delta$ and $\pm\frac{1}{2}\vec{\alpha}$ is not in Δ, then the only possible elements of $\Delta \cup \{\vec{0}\}$ proportional to $\vec{\alpha}$ are $\pm\vec{\alpha}$, $\pm 2\vec{\alpha}$, and $\vec{0}$.*
(3) *If $\vec{\alpha}$ is in Δ and $\vec{\beta} \in \Delta \cup \vec{0}$, then*

$$n(\vec{\alpha}, \vec{\beta}) := \frac{2\langle \vec{\beta}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} = 0, \pm 1, \pm 2, \pm 3 \text{ or } \pm 4\,, \tag{15}$$

and ± 4 can only occur if $\vec{\beta} = \pm 2\vec{\alpha}$.

Proof. (1) Consider

$$s_{\vec{\alpha}}(\vec{\alpha}) = \vec{\alpha} - \frac{2\langle \vec{\alpha}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle}\vec{\alpha} = -\vec{\alpha}\,. \tag{16}$$

By the definition of an abstract root system, the reflections map the set Δ to itself. Hence, if $\vec{\alpha} \in \Delta$, then $-\vec{\alpha} \in \Delta$.

(2) To prove the second property, we will use the fact that $n(\vec{\alpha}, \vec{\beta})$ must be an integer. Hence, if $k \in \mathbb{R}$,

$$\frac{2\langle k\vec{\alpha}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \in \mathbb{Z} \quad \text{and} \quad \frac{2\langle \vec{\alpha}, k\vec{\alpha} \rangle}{\langle k\vec{\alpha}, k\vec{\alpha} \rangle} \in \mathbb{Z}\,.$$

By the properties of the inner product, it follows that $2/k$ and $2k$ are both integers. We know that either $k = 0$ or $|k| \geq 1/2$ for $2k \in \mathbb{Z}$. We also know that $2/|k|$ can only be an integer larger than 4 if $|k| < 1/2$. Hence, it suffices to find the k that satisfy the equation $2/k = c$, where $c = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. We can rewrite this equation as $k = 2/c$ to see that $k = \{\pm 2, \pm 1, \pm 2/3, \pm 1/2\}$. Reject $k = \pm 2/3$, since $4/3 \notin \mathbb{Z}$, and $k = \pm 1/2$ by the statement of the theorem. Consequently, the only possible elements of $\Delta \cup \{\vec{0}\}$ proportional to $\vec{\alpha}$

are $\pm\vec{\alpha}$, $\pm 2\vec{\alpha}$, and $\vec{0}$.

(3) The third property is proved using the Cauchy-Schwarz inequality, which states that

$$\left|\langle\vec{\beta},\vec{\alpha}\rangle\right| \leq \|\vec{\alpha}\|\cdot\|\vec{\beta}\|\,. \tag{17}$$

We can rewrite (17) as

$$\left|\langle\vec{\beta},\vec{\alpha}\rangle\right| \leq \langle\vec{\alpha},\vec{\alpha}\rangle^{1/2}\cdot\langle\vec{\beta},\vec{\beta}\rangle^{1/2}\,.$$

Squaring both sides we notice that

$$\langle\vec{\beta},\vec{\alpha}\rangle^{2} \leq \langle\vec{\alpha},\vec{\alpha}\rangle\cdot\langle\vec{\beta},\vec{\beta}\rangle\,.$$

By the bilinearity of the inner product,

$$\langle\vec{\beta},\vec{\alpha}\rangle\cdot\langle\vec{\alpha},\vec{\beta}\rangle \leq \langle\vec{\alpha},\vec{\alpha}\rangle\cdot\langle\vec{\beta},\vec{\beta}\rangle\,.$$

It therefore follows that

$$\left|\frac{2\langle\vec{\beta},\vec{\alpha}\rangle}{\langle\vec{\alpha},\vec{\alpha}\rangle}\cdot\frac{2\langle\vec{\alpha},\vec{\beta}\rangle}{\langle\vec{\beta},\vec{\beta}\rangle}\right| \leq 4\,. \tag{18}$$

Once again, the fractions

$$\frac{2\langle\vec{\beta},\vec{\alpha}\rangle}{\langle\vec{\alpha},\vec{\alpha}\rangle} \quad\text{and}\quad \frac{2\langle\vec{\alpha},\vec{\beta}\rangle}{\langle\vec{\beta},\vec{\beta}\rangle}$$

must be integers. When taken together, their product must be less than or equal to four, so each of these fractions can only be 0, ±1, ±2, ±3, or ±4.

Suppose that

$$\frac{2\langle\vec{\beta},\vec{\alpha}\rangle}{\langle\vec{\alpha},\vec{\alpha}\rangle} = \pm 4\,.$$

In this case, equality holds in the Cauchy-Schwarz inequality, so $\vec{\alpha}$ and $\vec{\beta}$ are proportional. In addition, by (17),

$$\frac{2\langle\vec{\alpha},\vec{\beta}\rangle}{\langle\vec{\beta},\vec{\beta}\rangle} = \pm 1\,.$$

As a result, $2\langle\vec{\beta},\vec{\alpha}\rangle = 4\|\vec{\alpha}\|^2$ and $2\langle\vec{\alpha},\vec{\beta}\rangle = 2\|\vec{\beta}\|^2$, so $\|\vec{\beta}\| = 2\|\vec{\alpha}\|$. Since $\vec{\alpha}$ is proportional to $\vec{\beta}$, it is clear that $\vec{\beta} = \pm 2\vec{\alpha}$.

□

Property (3) of theorem 6 limits the magnitude of $n(\vec{\alpha},\vec{\beta})$. This leads us to consider the possible values of $n(\vec{\alpha},\vec{\beta})$ for the familiar, two dimensional root systems of type A_2, B_2, and $A_1 \oplus A_1$. Brute force calculations show that in each of these three cases, $n(\vec{\alpha},\vec{\beta})$ can only be 0, ±1, or ±2. We must now try to find abstract root systems in $V = \mathbb{R}^2$ which allow $n(\vec{\alpha},\vec{\beta})$

to equal ± 3 or ± 4. To accomplish this goal, we must first prove another theorem.

Theorem 7. *Let Δ be an abstract root system in V.*

(1) *If $\vec{\alpha}$ and $\vec{\beta}$ are in Δ, and $\langle\vec{\alpha},\vec{\beta}\rangle > 0$, then $\vec{\alpha}-\vec{\beta}$ is a root or 0. If $\langle\vec{\alpha},\vec{\beta}\rangle < 0$, then $\vec{\alpha}+\vec{\beta}$ is a root or 0.*
(2) *Let $\vec{\alpha}\in\Delta$ and $\vec{\beta}\in\Delta\cup\{\vec{0}\}$. If $\vec{\beta}+n\vec{\alpha},\vec{\beta}+(n+1)\vec{\alpha}\in\Delta\cup\{\vec{0}\}$, then $\vec{\beta}+(n+1)\vec{\alpha}$ must also be in $\Delta\cup\{\vec{0}\}$.*

Proof. (1) Consider the following two reflections:

$$s_{\vec{\alpha}}(\vec{\beta}) = \vec{\beta}-n(\vec{\alpha},\vec{\beta})\vec{\alpha} \tag{19}$$

$$s_{\vec{\beta}}(\vec{\alpha}) = \vec{\alpha}-n(\vec{\beta},\vec{\alpha})\vec{\beta} \tag{20}$$

For $\vec{\alpha}-\vec{\beta}$ to be a root, it suffices that $n(\vec{\beta},\vec{\alpha})=1$ or $n(\vec{\alpha},\vec{\beta})=1$. For $\vec{\alpha}+\vec{\beta}$ to be a root, it also suffices that $n(\vec{\beta},\vec{\alpha})=-1$ or $n(\vec{\alpha},\vec{\beta})=-1$. By equation (18), $\left|n(\vec{\alpha},\vec{\beta})\cdot n(\vec{\beta},\vec{\alpha})\right|\leq 4$. Hence, the following are the possible values for $n(\vec{\alpha},\vec{\beta})$ and $n(\vec{\beta},\vec{\alpha})$:

$n(\vec{\alpha},\vec{\beta})$	$n(\vec{\beta},\vec{\alpha})$
± 1	$\pm 1, \pm 2, \pm 3, \pm 4$
± 2	$\pm 1, \pm 2$
± 3	± 1
± 4	± 1

When $n(\vec{\alpha},\vec{\beta})=\pm 2$ and $n(\vec{\beta},\vec{\alpha})=\pm 2$, $\left|n(\vec{\alpha},\vec{\beta})\right|=\left|n(\vec{\beta},\vec{\alpha})\right|$. Hence $\langle\vec{\alpha},\vec{\alpha}\rangle=\langle\vec{\beta},\vec{\beta}\rangle$. We know by the previous theorem that $\vec{\alpha}$ and $\vec{\beta}$ are proportional. Thus, $\vec{\beta}$ must be $\pm\vec{\alpha}$. In every other case, either $n(\vec{\alpha},\vec{\beta})$ or $n(\vec{\beta},\vec{\alpha})$ must be ± 1. If $\langle\vec{\alpha},\vec{\beta}\rangle>0$, then $n(\vec{\alpha},\vec{\beta})>0$ and $n(\vec{\beta},\vec{\alpha})>0$. Therefore, the reflections in equations (19) and (20) yield either $\vec{\beta}-\vec{\alpha}$ or $\vec{\alpha}-\vec{\beta}$. If $\langle\vec{\alpha},\vec{\beta}\rangle<0$, then $n(\vec{\alpha},\vec{\beta})<0$ and $n(\vec{\beta},\vec{\alpha})<0$. Now, these reflections yield $\vec{\alpha}+\vec{\beta}$.

(2) We will prove the second statement by contradiction. Suppose that $\vec{\beta}+n\vec{\alpha},\vec{\beta}+(n+2)\vec{\alpha}\in\Delta\cup\{\vec{0}\}$, but $\beta+(n+1)\vec{\alpha}\notin\Delta\cup\{\vec{0}\}$. Hence, we assume that there is a gap in the set of elements of $\Delta\cup\{\vec{0}\}$ of the form $\vec{\beta}+n\vec{\alpha}$. We know that $\vec{\alpha}\in\Delta$, so by the first part of theorem 7, either

$$\vec{\beta}+(n+2)\vec{\alpha}-\vec{\alpha}=\vec{\beta}+(n+1)\vec{\alpha}\in\Delta\cup\{\vec{0}\} \tag{21}$$

if $\langle\vec{\beta}+(n+2)\vec{\alpha},\vec{\alpha}\rangle > 0$, or

$$\vec{\beta}+n\vec{\alpha}+\vec{\alpha}=\vec{\beta}+(n+1)\vec{\alpha}\in\Delta\cup\{\vec{0}\} \tag{22}$$

if $\langle\vec{\beta}+n\vec{\alpha},\vec{\alpha}\rangle < 0$. By simplifying these conditions, we observe that $\vec{\beta}+(n+1)\vec{\alpha}\in\Delta\cup\{\vec{0}\}$ if $\langle\vec{\beta},\vec{\alpha}\rangle > -(n+2)\langle\vec{\alpha},\vec{\alpha}\rangle$ or $\langle\vec{\beta},\vec{\alpha}\rangle < -n\langle\vec{\alpha},\vec{\alpha}\rangle$. These two conditions cover all possibilities for $\langle\vec{\alpha},\vec{\beta}\rangle$, so $\vec{\beta}+(n+1)\vec{\alpha}\in\Delta\cup\{\vec{0}\}$, which contradicts our original proposition.

□

We will conclude by taking advantage of the Euclidean geometry to describe the geometry of abstract root systems. Recall that for the standard inner product in $\mathbb{R}^n$, the number $\langle\vec{\alpha},\vec{\alpha}\rangle = \|\vec{\alpha}\|^2$ is the square of the length of the vector. Hence, $n(\vec{\alpha},\vec{\beta})$ can be written as

$$n(\vec{\alpha},\vec{\beta}) = 2\frac{\|\vec{\beta}\|}{\|\vec{\alpha}\|}\cos\phi\,, \tag{23}$$

where ϕ is the angle between $\vec{\alpha}$ and $\vec{\beta}$. Then we have

$$\left|n(\vec{\alpha},\vec{\beta})\cdot n(\vec{\beta},\vec{\alpha})\right| = 4\cos^2\phi\,.$$

By applying theorem 6, we can find all of the possible values for ϕ, as shown in the table below.

$n(\vec{\alpha},\vec{\beta})$	$n(\vec{\beta},\vec{\alpha})$	$\left\|n(\vec{\alpha},\vec{\beta})\cdot n(\vec{\beta},\vec{\alpha})\right\|$	$\cos\phi$	ϕ
0	0	0	0	90°
±1	$\pm1,\pm2,\pm3,\pm4$	$1,2,3,4$	$1/2, 1/\sqrt{2}, \sqrt{3}/2, 1$	$60^\circ, 45^\circ, 30^\circ, 0^\circ$
±2	$\pm1,\pm2$	$2,4$	$1/\sqrt{2}, 1$	$45^\circ, 0^\circ$
±3	±1	3	$\sqrt{3}/2$	30°
±4	±1	4	1	0°

Consequently, the angle ϕ between two nonproportional elements of an abstract root system can only be 30°, 45°, 60°, or 90°.

The relative lengths of any two vectors can also be predicted. Equation (23) also implies that

$$n(\vec{\beta},\vec{\alpha}) = 2\frac{\|\vec{\alpha}\|}{\|\vec{\beta}\|}\cos\phi\,,$$

so we will now calculate all possible ratios $\|\vec{\beta}\|/\|\vec{\alpha}\|$ and $\|\vec{\alpha}\|/\|\vec{\beta}\|$ of the roots for a fixed angle ϕ. The possible relative lengths are therefore those values that satisfy both ratios, as shown in the following table.

ϕ	$\|\vec{\beta}\|/\|\vec{\alpha}\|$	$\|\vec{\alpha}\|/\|\vec{\beta}\|$	relative length ≥ 1
30°	$1/\sqrt{3}, \frac{2}{3}\sqrt{3}, \sqrt{3}, \frac{4}{3}\sqrt{3}$	$\sqrt{3}, \frac{3}{2}\sqrt{3}, 1/\sqrt{3}, \frac{3}{4}\sqrt{3}$	$\sqrt{3}$
45°	$1/\sqrt{2}, \sqrt{2}, \frac{3}{2}\sqrt{2}, 2\sqrt{2}$	$\sqrt{2}, 1/\sqrt{2}, \frac{2}{3}\sqrt{2}, \frac{1}{2\sqrt{2}}$	$\sqrt{2}$
60°	$1, 2, 3, 4$	$1, 1/2, 1/3, 1/4$	1
90°	$1/2, 1, 3/2, 2$	$2, 1, 2/3, 1/2$	$1, 2$

We now have all the tools we need to describe all possible root systems in $V = \mathbb{R}^2$. We have already encountered three of them—$A_1 \oplus A_1$, A_2, and B_2. Recall that for $A_1 \oplus A_1$, 4 roots meet at 90° angles. For A_2 abstract root systems, 6 roots meet with 60° angles between adjacent roots, so the only relative length they can have is 1. For B_2 systems, 8 roots meet with 45° angles between adjacent ones, so the possible relative lengths are $1/\sqrt{2}$ for those vectors with 45° between them and 1 for those vectors with 90° between them. If the relative lengths of vectors that intersect at 45° angles in B_2 is instead $\sqrt{2}$, we have a fifth abstract root system, C_2. In effect, this is just a rotated version B_2. If we superimpose the B_2 and C_2 root systems, we get a fifth abstract root system in $\mathbb{R}^2$ known as BC_2. Finally, if we take the angle between 12 adjacent roots to be 30° apart, we see that we get relative length to be $\sqrt{3}$ between adjacent roots and 1 between alternating roots. We call this sixth root system G_2. The root systems C_2, BC_2, and G_2 are depicted below.

FIGURE 6. C_2 Abstract Root System

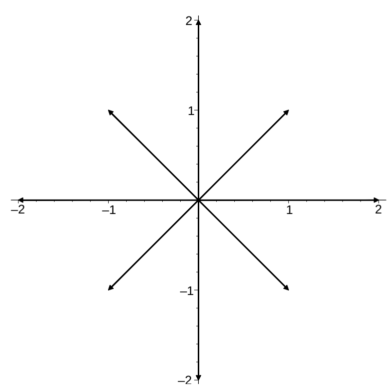


FIGURE 7. BC_2 Abstract Root System

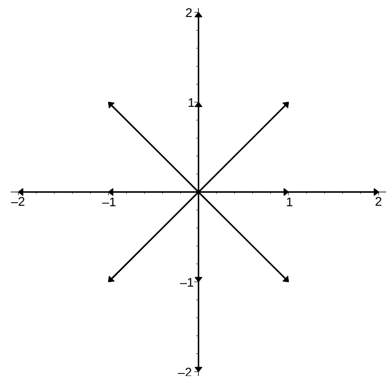
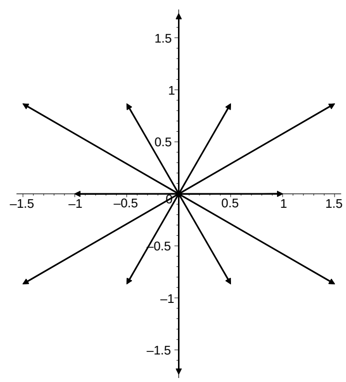


FIGURE 8. G_2 Abstract Root System



No other root systems can possibly exist in $\mathbb{R}^2$. Thus, maximum number of roots in any root system on $\mathbb{R}^2$ is 12.

References

[1] Anna Bergren, *Reflections in a Euclidean Space*, preprint, MIT, 2004.
[2] Wikipedia, *Root Systems*, <http://www.fact-index.com/r/o/root_system.html>.