18.04 Midterm 2 Review Session
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1. Harmonic functions
1.1. Real and imaginary parts of an analytic function are harmonic.
1.2. Viewing a harmonic function on a simply connected region as the real part of an analytic function.
1.3. Harmonic conjugates: finding harmonic conjugates; orthogonality of level curves of harmonic conjugates.
1.4. Mean value property.
1.5. Maximum and minimum principles.
1.6. Finding harmonic functions on wedges etc. with prescribed boundary conditions by "pulling back" a simpler harmonic function by an analytic map (Problem 4(a) on Homework 5, Problem 2 on Recitation 6).

2. 2D hydrodynamics
2.1. Stationary flows; incompressible (divergence free) flows; irrotational (curl free) flows. Expressing these assumptions in terms of properties of the velocity vector field.
2.2. Velocity fields associated to a complex potential function. Such velocity fields are automatically incompressible and irrotational.
2.3. Complex velocity and stream function associated to a complex potential function. The flow is along level curves of the stream function. Stagnation points.
2.4. Deriving a complex potential for an incompressible, irrotational flow on a simply connected region.
2.5. Examples of flows: uniform flow, linear source, double source, linear vortex, and combinations of these.

3. Taylor series
3.1. (Absolute) convergence of power series – radius of convergence; ratio test; root test; by comparison (see Problem 3(b), Problem Fun 4 on Homework 6).
3.2. Taylor's theorem for analytic functions.
3.3. Finding Taylor series explicitly for functions, either by directly using the formula, or (whenever possible) by using the known Taylor series for $\sin(x)$, $\cos(x)$, e^x, $(1+x)^{-1}$ etc.
3.4. Analytic functions on connected domains which are not identically 0 have isolated zeroes.
3.5. Power series solutions of differential equations by iteratively solving for coefficients.

4. Isolated singularities and Laurent series
4.1. Types of isolated singularities – removable singularities, poles (and order of a pole), essential singularities. Holomorphic and meromorphic functions.
4.2. Finding examples of functions with some prescribed isolated singularities.
4.3. Laurent series for functions which are analytic on an annulus. Analytic (regular) and principal (singular) part of Laurent series.
4.4. Finding Laurent series of functions using known series and algebraic manipulations (e.g. partial fractions).
4.5. Behavior of isolated singularities under basic algebraic operations like addition/multiplication/division.

5. Residues
5.1. Computing residues of functions using known series and algebraic manipulations (similar to 4.4 above).
5.2. Computing residues at a simple pole z_0 by multiplying by $(z - z_0)$ and then proceeding, for example, using L'Hopital's rule; computing the residue of $1/g(z)$ at z_0 where $g(z)$ has a simple pole at z_0.
5.3. Computing residues at a pole of order k, either directly from the series, or after multiplying by $(z - z_0)^k$ and then computing the appropriate derivative (suitably normalized).
5.4. Poles and residues of $\cot(z)$.
5.5. Cauchy's residue theorem.
5.6. Residue at ∞.

6. Computing definite integrals using the residue theorem
6.1. Integrals of functions that decay like $\frac{1}{|z|^\alpha}$ for some $\alpha > 1$ using the semicircular contour or possibly using a circular sector (as in Problem 2(c) on Homework 7).
6.2. Integrals of functions of the form $f(z)e^{iaz}$, where $f(z)$ decays like $1/|z|$ for large enough values of z using the appropriate box contour.
6.3. Evaluating trigonometric integrals from 0 to ∞ etc. by writing $\sin(x)$, $\cos(x)$ in terms of exponentials or (if appropriate) as the imaginary/real part of e^{ix}.
6.4. Trigonometric integrals from 0 to 2π etc. via the unit circle contour and the replacement $z = e^{i\theta}$.
6.5. Integrals for functions with branch cuts using keyhole contours. Be careful with signs here.
6.6. Integrals over portions of the circle around a pole.
6.7. Indented contour and principal values.

7. Fourier transform
7.1. Fourier transform and Fourier inversion formula.
7.2. Solving differential equations using the Fourier transform.