18.04 Recitation 8
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1.1 Show that if $g(z)$ has a simple zero at z_0, then $1/g(z)$ has a simple pole at z_0.

1.2. Show that $\mathrm{Res}(1/g, z_0) = 1/g'(z_0)$.

1.3. Let $f(z) = 1/\sin(z)$. Find all the poles, show that they are simple, and use the previous part to find the residues at these poles.

2.1. Let $p(z)$ and $q(z)$ be analytic at $z = z_0$. Assume $p(z_0) \neq 0$ and q has a simple zero at z_0. Show that $\mathrm{Res}_{z=z_0}(p(z)/q(z)) = p(z_0)/q'(z_0)$.

2.2. Let $f(z) = \cot(z)$. Find all the poles, show that they are simple, and use the previous part to the find residues at these poles.

3. By using the Taylor series of $\cos(z)$ and $\sin(z)$ around $z = 0$, compute the first few terms of the Laurent expansion of $\cot(z)$ around $z = 0$.

4. Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise.

4.1. Suppose that there is only one isolated singularity inside C at the point z_1. By using the extended version of Cauchy's theorem, show that $\int_C f(z)dz = 2\pi i \mathrm{Res}(f, z_1)$.

4.2. Suppose now that there are two isolated singularities inside C at the points z_1 and z_2. Again, by using the extended version of Cauchy's theorem, show that $\int_C f(z)dz = 2\pi i \left(\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2)\right)$.

4.3. Generalize the previous part to show that

$$\int_C f(z)dz = 2\pi i \left(\sum \text{residues of } f \text{ inside } C\right).$$

This is Cauchy's residue theorem.