18.04 Recitation 12
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1. Use Rouche's theorem to show that all 5 zeros of $z^5 + 3z + 1$ are inside the curve $C_2 := \{z : |z| = 2\}$.

Ans: Example 11.7 in the notes.

2. Use Rouche's theorem to show that $z + 3 + 2e^z$ has one zero in the left half plane.

Ans: Example 11.8 in the notes.

3. Use Rouche's theorem to give another proof of the fundamental theorem of algebra i.e. to show that $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has exactly n roots in the complex plane.

Ans: Theorem following Example 11.8 in the notes.

4. Let $G(z)$ be a meromorphic function, and let $H(z) := \frac{G(z)}{1+G(z)}$. For a closed curve γ such that $G\circ\gamma$ does not go through -1 and G does not have any poles on γ, show that $P_{H,\gamma} = P_{G,\gamma} + \mathrm{Ind}(G\circ\gamma, -1)$.

Ans: Note that $P_{H,\gamma} = Z_{1+G,\gamma}$, so it suffices to show that $Z_{1+G,\gamma} - P_{G,\gamma} = \mathrm{Ind}(G\circ\gamma, -1)$. Since $1 + G$ does not have zeros or poles on γ, the argument principle shows that

$$\begin{aligned} Z_{1+G,\gamma} - P_{1+G,\gamma} &= \mathrm{Ind}((1+G)\circ\gamma, 0) \\ &= \mathrm{Ind}(1 + G\circ\gamma, 0) \\ &= \mathrm{Ind}(G\circ\gamma, -1). \end{aligned}$$

Finally, note that $P_{1+G,\gamma} = P_{G,\gamma}$.

5. Consider the method of images if there is a source at $(0,0)$, and walls at $y = 1$ and $y = -1$. How many image sources do you need? What is the resulting complex potential?

Ans: We add an image source at $(0,2)$ to make the $y = 1$ wall a streamline. Then, we add image sources at $(0,-2)$ and $(0,-4)$ to make the $y = -1$ wall a streamline. However, when we do so, the $y = 1$ wall is no longer a streamline, so we need to add sources at $(0,4)$ and $(0,6)$. When we do this, the $y = -1$ wall is no longer a streamline, and so on. Continuing this process, we see that we need a source at $\pm(0,2n)$ for all $n \geq 1$, and the resulting complex potential is

$$\Phi(z) = \log(z) + \sum_{n\geq 1} \log(z + 2ni) + \sum_{n\leq -1} \log(z - 2ni).$$