Topic 13 Notes

Jeremy Orloff

13 Analytic continuation and the Gamma function

13.1 Introduction

In this topic we will look at the Gamma function. This is an important and fascinating function that generalizes factorials from integers to all complex numbers. We look at a few of its many interesting properties. In particular, we will look at its connection to the Laplace transform.

We will start by discussing the notion of analytic continuation. We will see that we have, in fact, been using this already without any comment. This was a little sloppy mathematically speaking and we will make it more precise here.

13.2 Analytic continuation

If we have an function which is analytic on a region A, we can sometimes extend the function to be analytic on a bigger region. This is called analytic continuation.

Example 13.1. Consider the function

$$F(z) = \int_0^\infty \mathrm{e}^{3t}\mathrm{e}^{-zt}\,dt. \tag{1}$$

We recognize this as the Laplace transform of $f(t) = \mathrm{e}^{3t}$ (though we switched the variable from s to z). The integral converges absolutely and F is analytic in the region $A = \{\mathrm{Re}(z) > 3\}$.

Can we extend $F(z)$ to be analytic on a bigger region B? That is, can we find a region B a function $\tilde{F}(z)$ such that

1. B contains A
2. $\tilde{F}(z)$ is analytic on B
3. $\tilde{F}(z)$ agrees with F on A, i.e. $\tilde{F}(z) = F(z)$ for $z \in A$.

Solution: Yes! We know that $F(z) = \dfrac{1}{z-3}$ –valid for any z in A. So we can define $\tilde{F}(z) = \dfrac{1}{z-3}$ for any z in $B = \mathbf{C} - \{3\}$.

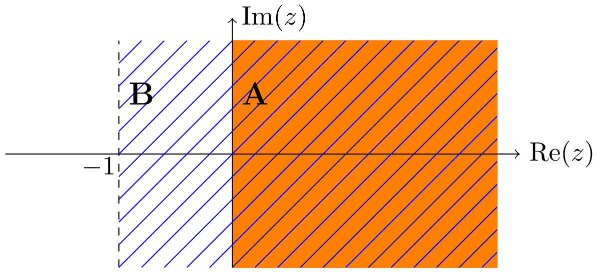
We say that we have analytically continued F on A to $\tilde{F}$ on B.

Note. Usually we don't rename the function. We would just say $F(z)$ defined by Equation 1 can be continued to $F(z) = \dfrac{1}{z-3}$ on B.

Definition. Suppose $f(z)$ is analytic on a region A. Suppose also that A is contained in a region B. We say that f can be analytically continued from A to B if there is a function $\tilde{f}(z)$ such that

1. $\tilde{f}(z)$ is analytic on B.
2. $\tilde{f}(z) = f(z)$ for all z in A.

As noted above, we usually just use the same symbol f for the function on A and its continuation to B.



The region $A = \text{Re}(z) > 0$ is contained in $B = \text{Re}(z) > -1$.

Note. We used analytic continuation implicitly in, for example, the Laplace inversion formula involving residues of $F(s) = \mathcal{L}(f; s)$. Recall that we wrote that for $f(t) = e^{3t}$, $F(s) = \frac{1}{s-3}$ and

$$f(t) = \sum \text{residues of } F.$$

As an integral, $F(s)$ was defined for $\text{Re}(s) > 3$, but the residue formula relies on its analytic continuation to $\mathbf{C} - \{3\}$.

13.2.1 Analytic continuation is unique

Theorem 13.2. Suppose f, g are analytic on a connected region A. If $f = g$ on an open subset of A then $f = g$ on all of A.

Proof. Let $h = f - g$. By hypothesis $h(z) = 0$ on an open set in A. Clearly this means that the zeros of h are not isolated. Back in Topic 7 we showed that for analytic h on a connected region A either the zeros are isolated or else h is identically zero on A. Thus, h is identically 0, which implies $f = g$ on A.

Corollary. There is at most one way to analytically continue a function from a region A to a connected region B.

Proof. Two analytic continuations would agree on A and therefore must be the same.

Extension. Since the proof of the theorem uses the fact that zeros are isolated, we actually have the stronger statement: if f and g agree on a nondiscrete subset of A then they are equal. In particular, if f and g are two analytic functions on A and they agree on a line or ray in A then they are equal.

Here is an example that shows why we need A to be connected in Theorem 13.2.

Example 13.3. Suppose A is the plane minus the real axis. Define two functions on A as follows.

$$f(z) = \begin{cases} 1 & \text{for } z \text{ in the upper half-plane} \\ 0 & \text{for } z \text{ in the lower half-plane} \end{cases}$$

$$g(z) = \begin{cases} 1 & \text{for } z \text{ in the upper half-plane} \\ 1 & \text{for } z \text{ in the lower half-plane} \end{cases}$$

Both f and g are analytic on A and agree on an open set (the upper half-plane), but they are not the same function.

Here is an example that shows a little care must be taken in applying the corollary.

Example 13.4. Suppose we define f and g as follows

$$\begin{aligned} f(z) &= \log(z) \text{ with } 0 < \theta < 2\pi \\ g(z) &= \log(z) \text{ with } -\pi < \theta < \pi \end{aligned}$$

Clearly f and g agree on the first quadrant. But we can't use the theorem to conclude that $f = g$ everywhere. The problem is that the regions where they are defined are different. f is defined on $\mathbf{C}$ minus the positive real axis, and g is defined on $\mathbf{C}$ minus the negative real axis. The region where they are both defined is $\mathbf{C}$ minus the real axis, which is not connected.

Because they are both defined on the upper half-plane, we can conclude that they are the same there. (It's easy to see this is true.) But (in this case) being equal in the first quadrant doesn't imply they are the same in the lower half-plane.

13.3 Definition and properties of the Gamma function

Definition. The Gamma function is defined by the integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, dt \tag{2}$$

The integral converges absolutely for $\mathrm{Re}(z) > 0$.

Properties

1. $\Gamma(z)$ is defined and analytic in the region $\mathrm{Re}(z) > 0$.
2. $\Gamma(n+1) = n!$, for integers $n \geq 0$.
3. $\Gamma(z+1) = z\Gamma(z)$ (functional equation)
   This property and Property 2 characterize the factorial function. Thus, $\Gamma(z)$ generalizes $n!$ to complex numbers z. Some authors will write $\Gamma(z+1) = z!$.
4. $\Gamma(z)$ can be analytically continued to be meromorphic on the entire plane with simple poles at $0, -1, -2 \ldots$. The residues are
   $$\mathrm{Res}(\Gamma, -m) = \frac{(-1)^m}{m!}$$
5. $\Gamma(z) = \left[z\mathrm{e}^{\gamma z} \prod_1^\infty \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n} \right]^{-1}$, where γ is Euler's constant
   $$\gamma = \lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} - \log(n) \approx 0.577$$
   This property uses an infinite product. Unfortunately we won't have time, but infinte products represent an entire topic on their own. Note that the infinite product makes the positions of the poles of Γ clear.
6. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$
   With Property 5 this gives a product formula for $\sin(\pi z)$.

7. $\Gamma(z+1) \approx \sqrt{2\pi} z^{z+1/2} e^{-z}$ for $|z|$ large, $\mathrm{Re}(z) > 0$.
   In particular, $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$. (Stirling's formula)
8. $2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z)$ (Legendre duplication formula)

Notes. These are just some of the many properties of $\Gamma(z)$. As is often the case, we could have chosen to define $\Gamma(z)$ in terms of some of its properties and derived Equation 2 as a theorem.

We will prove (some of) these properties below.

Example 13.5. Use the properties of Γ to show that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$.

Solution: From Property 2 we have $\Gamma(1) = 0! = 1$. The Legendre duplication formula with $z = 1/2$ then shows

$$2^0\Gamma\left(\frac{1}{2}\right)\Gamma(1) = \sqrt{\pi}\Gamma(1) \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Now, using the functional equation Property 3 we get

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

13.4 Connection to Laplace

Claim. For $\mathrm{Re}(z) > 1$ and $\mathrm{Re}(s) > 0$, $\mathcal{L}(t^{z-1}; s) = \dfrac{\Gamma(z)}{s^z}$.

Proof. By definition $\mathcal{L}(t^{z-1}; s) = \displaystyle\int_0^\infty t^{z-1} e^{-st}\,dt$. It is clear that if $\mathrm{Re}(z) > 1$, then the integral converges absolutely for $\mathrm{Re}(s) > 0$.

Let's start by assuming that $s > 0$ is real. Use the change of variable $\tau = st$. The Laplace integral becomes

$$\int_0^\infty t^{z-1} e^{-st}\,dt = \int_0^\infty \left(\frac{\tau}{s}\right)^{z-1} e^{-\tau}\frac{d\tau}{s} = \frac{1}{s^z}\int_0^\infty \tau^{z-1}e^{-\tau} = \frac{\Gamma(z)}{s^z}\,d\tau.$$

This shows that $\mathcal{L}(t^{z-1}; s) = \dfrac{\Gamma(z)}{s^z}$ for s real and positive. Since both sides of this equation are analytic on $\mathrm{Re}(s) > 0$, the extension to Theorem 13.2 guarantees they are the same.

Corollary. $\Gamma(z) = \mathcal{L}(t^{z-1}; 1)$. (Of course, this is also clear directly from the definition of $\Gamma(z)$ in Equation 2.

13.5 Proofs of (some) properties of Γ

Property 1. This is clear since the integral converges absolutely for $\mathrm{Re}(z) > 0$.

Property 2. We know (see the Laplace table) $\mathcal{L}(t^n; s) = \dfrac{n!}{s^{n+1}}$. Setting $s = 1$ and using the corollary to the claim above we get

$$\Gamma(n+1) = \mathcal{L}(t^n; 1) = n!.$$

(We could also prove this formula directly from the integral definition of of $\Gamma(z)$.)

Property 3. We could do this relatively easily using integration by parts, but let's continue using the Laplace transform. Let $f(t) = t^z$. We know

$$\mathcal{L}(f, s) = \frac{\Gamma(z+1)}{s^{z+1}}$$

Now assume $\mathrm{Re}(z) > 0$, so $f(0) = 0$. Then $f' = zt^{z-1}$ and we can compute $\mathcal{L}(f'; s)$ two ways.

$$\begin{aligned}
\mathcal{L}(f'; s) &= \mathcal{L}(zt^{z-1}; s) = \frac{z\Gamma(z)}{s^z} \\
\mathcal{L}(f'; s) &= s\mathcal{L}(t^z; s) = \frac{\Gamma(z+1)}{s^z}
\end{aligned}$$

Comparing these two equations we get property 3 for $\mathrm{Re}(z) > 0$.

Property 4. We'll need the following notation for regions in the plane.

$$\begin{aligned}
B_0 &= \{\mathrm{Re}(z) > 0\} \\
B_1 &= \{\mathrm{Re}(z) > -1\} - \{0\} \\
B_2 &= \{\mathrm{Re}(z) > -2\} - \{0, -1\} \\
B_n &= \{\mathrm{Re}(z) > -n\} - \{0, -1, \dots, -n+1\}
\end{aligned}$$

So far we know that $\Gamma(z)$ is defined and analytic on B_0. Our strategy is to use Property 3 to analytically continue Γ from B_0 to B_n. Along the way we will compute the residues at 0 and the negative integers.

Rewrite Property 3 as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{3}$$

The right side of this equation is analytic on B_1. Since it agrees with $\Gamma(z)$ on B_0 it represents an analytic continuation from B_0 to B_1. We easily compute

$$\mathrm{Res}(\Gamma, 0) = \lim_{z\to 0} z\Gamma(z) = \Gamma(1) = 1.$$

Similarly, Equation 3 can be expressed as $\Gamma(z+1) = \dfrac{\Gamma(z+2)}{z+1}$. So,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{(z+1)z} \tag{4}$$

The right side of this equation is analytic on B_2. Since it agrees with Γ on B_0 it is an analytic continuation to B_2. The residue at -1 is

$$\mathrm{Res}(\Gamma, -1) = \lim_{z\to -1} (z+1)\Gamma(z) = \frac{\Gamma(1)}{-1} = -1.$$

We can iterate this procedure as far as we want

$$\Gamma(z) = \frac{\Gamma(z+m+1)}{(z+m)(z+m-1)+\ldots+(z+1)z} \tag{5}$$

The right side of this equation is analytic on $\boldsymbol{B}_{m+1}$. Since it agrees with Γ on $\boldsymbol{B}_0$ it is an analytic continuation to $\boldsymbol{B}_{m+1}$. The residue at $-m$ is

$$\text{Res}(\Gamma, -m) = \lim_{z \to -m} (z+m)\Gamma(z) = \frac{\Gamma(1)}{(-1)(-2)\dots(-m)} = \frac{(-1)^m}{m!}.$$

We'll leave the proofs of Properties 5-8 to another class!