18.04 Practice problems for final exam, Spring 2018 Solutions

On the final exam you will be given a copy of the Laplace table posted with these problems.

Problem 1.
Which of the following are meromporphic in the whole plane.

(a) z^5

(b) $z^{5/2}$

(c) $e^{1/z}$

(d) $1/\sin(z)$.

answers: Meromorphic means analytic except for poles of *finite* order.

(a) Yes, this is entire.

(b) No, this requires a branch cut in the plane to define a region where it's analytic.

(c) No, the singularity at $z = 0$ is an essential singularity, not a finite pole.

(d) Yes, $\sin(z)$ has simple zeros at $n\pi$ for all integers n. So $1/\sin(z)$ has simple poles at these points.

Problem 2.
(a) *Let* $f(z) = \dfrac{(z-2)^2 z^3}{(z+5)^3(z+1)^3(z-1)^4}$. *Compute* $\displaystyle\int_{|z|=3} \frac{f'(z)}{f(z)}\, dz$

(b) *Find the number of roots of* $g(z) = 6z^4 + z^3 - 2z^2 + z - 1 = 0$ *in the unit disk.*

(c) *Suppose* $f(z)$ *is analytic on and inside the unit circle. Suppose also that* $|f(z)| < 1$ *for* $|z| = 1$. *Show that* $f(z)$ *has exactly one fixed point* $f(z_0) = z_0$ *inside the unit circle.*

(d) *True or false: Suppose* $f(z)$ *is analytic on and inside a simple closed curve* γ. *If* f *has* n *zeros inside* γ *then* $f'(z)$ *has* $n-1$ *zeros inside* γ.

answers: **(a)** By the argument principle the $\displaystyle\int_\gamma \frac{f'}{f}\, dz = 2\pi i(Z_{f,\gamma} - P_{f,\gamma}$. In this case, the zeros of f inside γ are 2, 0 of order 2 and 3 respectively. The poles inside γ are -1 and 1 of order 3 and 4 respectively. So, the integral equals

$$2\pi i(2 + 3 - 3 - 4) = -4\pi i.$$

(b) On the unit circle $|z^3 - 2z^2 + z - 1| < 5$ and $|6z^4| = 6$. Therefore by Rouche's theorem the number of zeros of $g(z)$ inside the unit circle is equal to the number of zeros of $6z^4$, i.e. 4.

(c) Let $g(z) = f(z) - z$. We want to show g has exactly one root inside the unit circle. We know $|f(z)| < |-z| = 1$ on the unit circle. So by Rouche's theorem $g(z)$ and $-z$ have the same number of zeros in the unit disk. That is, they both have exactly one such zero. QED.

(d) False. Consider $f(z) = e^z - 1$. This has 3 zeros inside the circle $|z| = 3\pi$ $(0, \pm 2\pi)$. But $f'(z) = e^z$ has no zeros.

Problem 3.
Let $A = \{z \,|\, 0 \le \text{Re}(z) \le \pi/2, \text{Im}(z) \ge 0$.

Let B = the first quadrant/

Show that $f(z) = \sin(z)$ maps A conformally onto B

answers: **(a)** You should supply a picture of the regions A and B and develop a picture tracking the argument we give. We see where f maps the boundary of A. The boundary of A has 3 pieces:

Piece 1: $z = iy$, with $y \geq 0$. On this piece

$$\sin(z) = \frac{e^{-y} - e^{y}}{2i} = \frac{(e^{y} - e^{-y})}{2}\, i$$

So, the image of piece 1 is the positive imaginary axis.

Piece 2: $z = x$, with $0 \leq x \leq \pi/2$. On this piece $\sin(z) = \sin(x)$, so the image runs from 0 to 1 along the real axis.

Piece 3: $z = \pi/2 + iy$, with $y \geq 0$. On this piece

$$\sin(z) = \frac{e^{-y+\pi i/2} - e^{y-\pi i/2}}{2i} = \frac{(ie^{-y} + ie^{-y})}{2i} = \frac{e^{-y} + e^{y}}{2} = \cosh(y).$$

So, the image of piece 3 is the real axis greater than 1.

We have shown that $f(z)$ maps the boundary of A to the boundary of B.

To see that A is mapped to B it's enough to verify that one point inside A is mapped to a point inside B. There are lots of ways to do this. Here's one. We know

$$\sin(x + iy) = \frac{e^{-y+ix} - e^{y-ix}}{2i}.$$

Pick $x = \pi/4$ and y so large that e^{-y} is very tiny. Then

$$\sin(x + iy) \approx -e^{y}e^{-ix}2i = -e^{y}\frac{\sqrt{2}/2 - i\sqrt{2}/2}{2i} = e^{y}\frac{\sqrt{2} + i\sqrt{2}}{4}$$

This last value is clearly in the first quadrant, i.e inside B.
