Part I Problems and Solutions

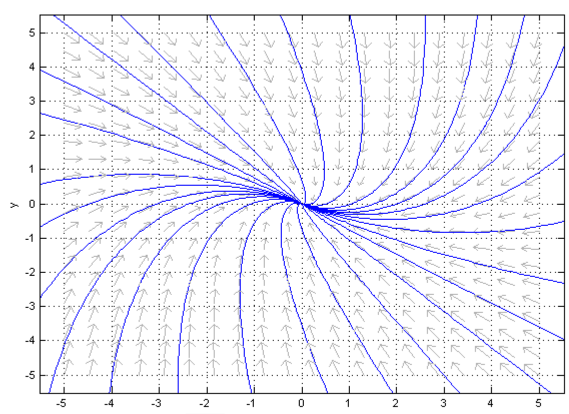
Problem 1: *Give the general solution to the DE system* $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \mathbf{x}$ *and also give its phase-plane picture (i.e its direction field graph together with a few typical solution curves).*

Solution: Characteristic equation $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \to$ repeated root $\lambda = -3$.
The single eigenvector $\mathbf{v}$ and a generalized eigenvector $\mathbf{w}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$, and the scalar component functions $x_1(t), x_2(t)$ of the general solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the form
$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(\mathbf{v}t + \mathbf{w})e^{\lambda t}$
of the given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are as follows:

Eigenvector: $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Generalized eigenvector: $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, $x_1(t) = (c_1 + c_2 + c_2 t)e^{-3t}$ and $x_2(t) = (-c_1 - c_2 t)e^{-3t}$.



Problem 2: *For each of the following linear systems, carry out the graphing program laid out in this session, that is:*

(i) *find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;*

(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing t; then draw some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

a) $x' = 2x - 3y, y' = x - 2y$

b) $x' = 2x, y' = 3x + y$

c) $x' = -2x - 2y, y' = -x - 3y$

d) $x' = x - 2y, y' = x + y$

e) $x' = x + y, y' = -2x - y$

Solution: Let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ throughout, and M be such that $\vec{x}'(t) = Mx(t)$. Let M have eigenvalues λ_1, λ_2, with corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. The general solution is thus

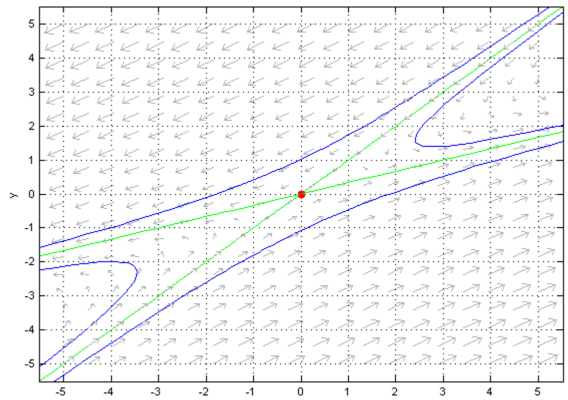
$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

a) $M = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$, with eigenvalues ± 1 and eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The system has a critical point at $(0,0)$ which is a saddle point.

For $c_1 = 0$ and as $t \to \infty$, $\vec{x}(t) = c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

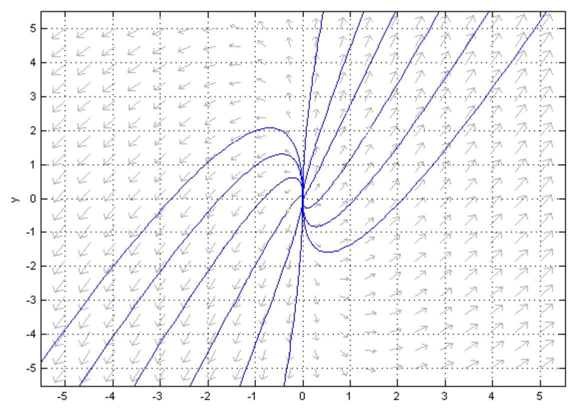
Similarly, for $c_2 = 0$ and $t \to -\infty$, $\vec{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus the behavior near the saddle point looks like



b) $M = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$, with eigenvalues $2, 1$ and eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The system has an unstable node at $(0, 0)$.
As $t \to -\infty$ all trajectories go to $\vec{0}$.

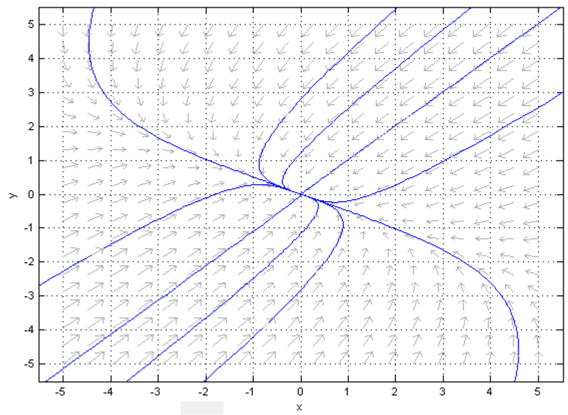
Thus the behavior near the node looks like



For $t \to -\infty$, $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$ is dominant term, so the solutions are near the y-axis. For $t \to \infty$, $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^2 t$ dominates, so solutions are parallel to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

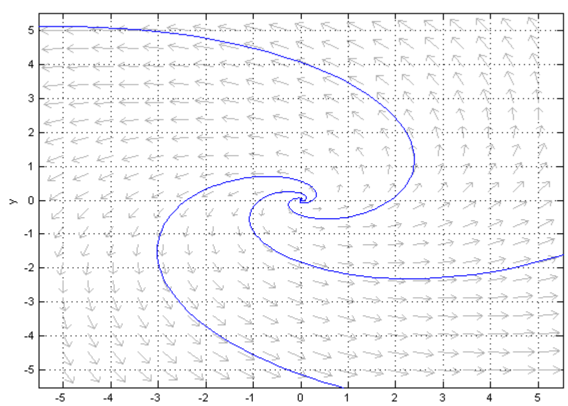
c) $M = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$, with eigenvalues $-4, -1$ and eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The system has an asymptotically unstable node at (0,0). As $t \to \infty$, all trajectories go to $\vec{0}$. The behavior near the origin looks like:

For $t \to -\infty$, $\begin{bmatrix}1\\1\end{bmatrix} e^{-4t}$ dominates, so solutions are parallel to $\begin{bmatrix}1\\1\end{bmatrix}$; for $t \to \infty$, $\begin{bmatrix}2\\-1\end{bmatrix} e^{-t}$ dominates, so solutions come in to the origin asymptotic to the line with direction vector $\begin{bmatrix}2\\-1\end{bmatrix}$.



d) $M = \begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}$, eigenvalues $1 \pm i\sqrt{2}$. The system then has an unstable spiral around (0,0).

Near $y = 0$, $x' \approx x$, so x is increasing where the spiral cuts the positive x-axis. As y increases, so does e^t, so the spiral is outwards from the origin.



e) $M = \begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}$. Eigenvalues are ± 1, pure imaginary, so the system has a stable center. The curves are ellipses, since $\frac{dy}{dx} = \frac{-2x-y}{x+y}$ which integrates easily after cross-multiplying to $2x^2 + 2xy + y^2 = c$.

Direction of motion: For instance, at $(1, 0)$ the vector field is $x' = 1$, $y' = -2$, so motion is clockwise.