The Exponential Matrix

The work in the preceding note with fundamental matrices was valid for any linear homogeneous square system of ODE's,

$$\mathbf{x}' = A(t)\,\mathbf{x}.$$

However, if the system has *constant coefficients*, i.e., the matrix A is a constant matrix, the results are usually expressed by using the exponential matrix, which we now define.

Recall that if x is any real number, then

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots . \tag{1}$$

Definition 3 Given an $n \times n$ constant matrix A, the **exponential matrix** e^A is the $n \times n$ matrix defined by

$$e^A = I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots . \tag{2}$$

Each term on the right side of (2) is an $n \times n$ matrix adding up the ij-th entry of each of these matrices gives you an infinite series whose sum is the ij-th entry of e^A. (The series always converges.)

In the applications, an independent variable t is usually included:

$$e^{At} = I + A\,t + A^2\,\frac{t^2}{2!} + \ldots + A^n\,\frac{t^n}{n!} + \ldots . \tag{3}$$

This is not a new definition, it's just (2) above applied to the matrix $A\,t$ in which every element of A has been multiplied by t, since for example

$$(At)^2 = At \cdot At = A \cdot A \cdot t^2 = A^2 t^2.$$

Try out (2) and (3) on these two examples (the second is very easy, since it is not an infinite series).

Example 3A. Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.   Show: $e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$;   and $e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

Example 3B. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show: $e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

What's the point of the exponential matrix? The answer is given by the theorem below, which says that the exponential matrix provides a royal road to the solution of a square system with constant coefficients: no eigenvectors, no eigenvalues, you just write down the answer!

Theorem 3 *Let A be a square constant matrix. Then*

(1)   *(a) $e^{At} = \widetilde{\Phi}_0(t)$, the normalized fundamental matrix at 0;*

(2)   *(b) the unique solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x} = e^{At}\mathbf{x}_0$.*

Proof. Recall that in the previous note we saw that if $\widetilde{\Phi}_0(t)$ is the normalized fundamental matrix then

The solution to the IVP : $\mathbf{x}' = A(t)\,\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = \widetilde{\Phi}_0(t)\mathbf{x}_0$.

$$(4)$$

Statement (2) follows immediately from (1), in view of (4).

We prove (1) is true by using the fact that if $t_0 = 0$ then the normalized fundamental matrix has $\Phi(0) = I$. Letting $\Phi = e^{At}$, we must show $\Phi' = A\Phi$ and $\Phi(0) = I$.

The second of these follows from substituting $t = 0$ into the infinite series definition (3) for e^{At}.

To show $\Phi' = A\Phi$, we assume that we can differentiate the series (3) term-by-term; then we have for the individual terms

$$\frac{d}{dt} A^n \frac{t^n}{n!} = A^n \cdot \frac{t^{n-1}}{(n-1)!},$$

since A^n is a constant matrix. Differentiating (3) term-by-term then gives

$$\begin{aligned} \frac{d\Phi}{dt} = \frac{d}{dt} e^{At} &= A + A^2 t + \ldots + A^n \frac{t^{n-1}}{(n-1)!} + \ldots \\ &= A\, e^{At} = A\,\Phi. \end{aligned} \qquad (5)$$

Calculation of e^{At}.

The main use of the exponential matrix is in Theorem 3 — writing down explicitly the solution to an IVP. If e^{At} has to be calculated for a specific system, several techniques are available.

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a) In simple cases, it can be calculated directly as an infinite series of matrices.

b) It can always be calculated, according to Theorem 3, as the normalized fundamental matrix $\tilde{\Phi}_0(t)$, using (11): $\tilde{\Phi}_0(t) = \Phi(t)\Phi(0)^{-1}$.

c) A third technique uses the exponential law

$$e^{(B+C)t} \;=\; e^{Bt}e^{Ct}, \qquad \text{valid if } BC = CB. \tag{6}$$

To use it, one looks for constant matrices B and C such that

$$A = B + C, \qquad BC = CB, \qquad e^{Bt} \text{ and } e^{Ct} \text{ are computable;} \qquad (7)$$

then

$$e^{At} \;=\; e^{Bt}e^{Ct}. \tag{8}$$

Example 3C.   Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.   Solve $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, using e^{At}.

Solution. We set $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;   then (7) is satisfied, and

$$e^{At} \;=\; \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

by (8) and Examples 3A and 3B. Therefore, by Theorem 3 (2), we get

$$\mathbf{x} \;=\; e^{At}\mathbf{x}_0 \;=\; e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{2t}\begin{pmatrix} 1 + 2t \\ 2 \end{pmatrix}.$$

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18.03SC Differential Equations
Fall 2011

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