Partial Fractions: Undetermined Coefficients

1. Introduction

Not every $F(s)$ we encounter is in the Laplace table. Partial fractions is a method for re-writing $F(s)$ in a form suitable for the use of the table.

In this note we will run through the various cases encountered when we apply the method of partial fractions decomposition to a rational function. In the next note we will learn the Heaviside cover-up method, which simplifies some of the algebra.

Note: We use the term undetermined coefficients in the same way it was use when solving an ODE with polynomial input. It involves setting a polynomial with unknown coefficients equal to a known polynomial and solving for the unknown coefficients by equating them with the known ones.

2. Rational Functions

A **rational function** is one that is the ratio of two polynomials. For example
$$\frac{s+1}{s^2+7s+9} \quad \text{and} \quad \frac{s^2+7s+9}{s+1}$$
are both rational functions.

A rational function is called **proper** if the degree of the numerator is strictly smaller than the degree of the denominator; in the examples above, the first is proper while the second is not.

Long-division: Using long-division we can always write an improper rational function as a polynomial plus a proper rational function. The partial fraction decomposition only applies to proper functions.

Example 1. Use long-division to write $\dfrac{s^3+2s+1}{s^2+s-2}$ as a the sum of a polynomial and a proper rational function.

Solution.

$$
\begin{array}{r|lllll}
 & s-1 & & & \\
\hline
s^2+s-2 & s^3 & + & 2s & +1 \\
 & s^3 & +s^2 & -2s & \\
\hline
 & & -s^2 & +4s & +1 \\
 & & -s^2 & -s & +2 \\
\hline
 & & & 5s & -1
\end{array}
$$

Therefore,
$$\frac{s^3 + 2s + 1}{s^2 + s - 2} = s - 1 + \frac{5s - 1}{s^2 + s - 2}.$$

3. Linear Factors

Here we assume the denominator factors in distinct linear factors. We start with a simple example. We will explain the general principle immediately afterwords.

Example 2. Decompose $R(s) = \dfrac{s - 3}{(s - 2)(s - 1)}$ using partial fractions. Use this to find $\mathcal{L}^{-1}(R(s))$.

Solution.
$$\frac{s - 3}{(s - 2)(s - 1)} = \frac{A}{s - 2} + \frac{B}{s - 1}.$$

Multiplying both sides by the denominator on the left gives
$$s - 3 = A(s - 1) + B(s - 2) \tag{1}$$

The sure algebraic way is to expand out the right hand side and equate the coefficients with those of the polynomial on the left.

$$s - 3 = (A + B)s + (-A - 2B) \;\Rightarrow\; \begin{cases} \text{coeff. of } s: & 1 \;=\; A + B \\ \text{coeff. of } 1: & -3 \;=\; -A - 2B \end{cases}$$

We solve this system of equations to find the undetermined coefficients A and B: $A = -1, \; B = 2$.
Answer: $R(s) = -1/(s - 2) + 2/(s - 1)$.
Table lookup then gives $\mathcal{L}^{-1}(R(s)) = -e^{2t} + 2e^t$.

Note: In this example it would have been easier to plug the roots of each factor into equation (1). When you do this every term except one becomes 0.
Plug in $s = 1 \;\Rightarrow\; -2 = B(-1) \Rightarrow B = 2$
Plug in $s = 2 \;\Rightarrow\; -1 = A(1) \Rightarrow A = -1$.

In general, if $P(s)/Q(s)$ is a proper rational function and $Q(s)$ factors into distinct linear factors $Q(s) = (s - a_1)(s - a_2) \cdots (s - a_n)$ then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2} + \cdots + \frac{A_n}{s - a_n}.$$

The proof of this is not hard, but we will not give it. Remember you must have a *proper* rational function and each of the factors must be distinct. Repeated factors are discussed below.

Example 3. Use partial fractions to find $\mathcal{L}^{-1}\left(\dfrac{3}{s^3 - 3s^2 - s + 3}\right)$.

Solution. The hardest part of this problem is to factor the denominator. For higher order polynomials this might be impossible. In this case you can check

$$s^3 - 3s^2 - s + 3 = (s - 1)(s + 1)(s - 3).$$

The partial fractions decomposition is

$$\frac{3}{(s-1)(s+1)(s-3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s-3}.$$

Multiplying through by the denominator gives

$$3 = A(s+1)(s-3) + B(s-1)(s-3) + C(s-1)(s+1).$$

Plugging in $s = 1$ gives $A = -3/4$, likewise $s = -1$ gives $B = 3/2$ and $s = 3$ gives $C = -3/4$.
Our answer is

$$\mathcal{L}^{-1}\left(\frac{3}{s^3 - 3s^2 - s + 3}\right) = Ae^t + Be^{-t} + Ce^{3t} = -\frac{2}{4}e^t + \frac{3}{2}e^{-t} - \frac{3}{4}e^{3t}.$$

4. Quadratic Factors

If the denominator has quadratic factors then, the numerator in the partial fraction decomposition will be a linear term instead of a constant.

Example 4. Find $\mathcal{L}^{-1}\left(\dfrac{s-1}{(s+1)(s^2+4)}\right)$.

Solution. This is a proper rational function so

$$\frac{s-1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}. \tag{2}$$

Notice the quadratic factor gets a linear term in the numerator. Notice also that the number of unknown coefficients is the same as the degree of the denominator in the original fraction.

From (2) we can write

$$\mathcal{L}^{-1}\left(\frac{s-1}{(s+1)(s^2+4)}\right) = Ae^{-t} + B\cos(2t) + \frac{C}{2}\sin(2t).$$

3

All that's left is to do some algebra to find the coefficients Muliplying (2) through by the denominator gives

$$s - 1 = A(s^2 + 4) + (Bs + C)(s + 1) = (A + B)s^2 + (B + C)s + (4A + C).$$

Equate the coefficients on both sides:

$$
\begin{aligned}
s^2: & \quad 0 = A + B \\
s: & \quad 1 = B + C \\
s^2: & \quad -1 = 4A + C
\end{aligned}
$$

Solving, we get $A = -2/5, \ B = 2/5, \ C = 3/5$.

Example 5. Don't be fooled by quadratic terms that factor into linear ones.

$$\frac{1}{(s + 1)(s^2 - 4)} = \frac{1}{(s + 1)(s + 2)(s - 2)} = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s - 2}.$$

Example 6. Don't forget that the rational function must be proper. For example, decompose $\dfrac{s^3 + 2s + 1}{s^2 + s - 2}$ using partial fractions.

Solution. First, we must use long-division to make this proper. From example (1) we have

$$\frac{s^3 + 2s + 1}{s^2 + s - 2} = s - 1 + \frac{5s - 1}{s^2 + s - 2} = s - 1 + \frac{5s - 1}{(s + 2)(s - 1)} = s - 1 + \frac{A}{s + 2} + \frac{B}{s - 1}.$$

Solving for the undetermined coefficients gives $A = 11/3, \ B = 4/3$.

5. Repeated Linear Factors

For repeated linear factors we need one partial fraction term for each power of the factor as illustrated by the following example.

Example 7. Find $\mathcal{L}^{-1}\left(\dfrac{2s}{s^3(s + 1)^2(s + 2)}\right)$ using partial fractions.

Solution.

$$\frac{2s}{s^3(s + 1)^2(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s + 1} + \frac{E}{(s + 1)^2} + \frac{F}{s + 2}$$

Here the denominator has a linear factor s repeated three times (term s^3), and a linear factor $(s + 1)$ repeated twice (term $(s + 1)^2$); hence three partial

fractions are associated with the first, while two are associated with the latter. The term (s+2) which is not repeated leads to one partial fraction as previously seen. You can check that the coefficients are

$$A = -5/2, \ B = 1, \ C = 0, \ D = 2, \ E = 2, \ F = 1/2.$$

Using the s-shift rule we have $\mathcal{L}^{-1}(1/(s+1)^2) = te^{-t}$. Thus,

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2s}{s^3(s+1)^2(s+2)}\right) &= A + Bt + \frac{C}{2}t^2 + De^{-t} + Ete^{-t} + Fe^{-2t} \\
&= -\frac{5}{2} + t + 2e^{-t} + 2te^{-t} + \frac{1}{2}e^{-2t}.
\end{aligned}
$$

6. Repeated Quadratic Factors

Just like repeated linear factors, quadratic factors have one term for each power of the factor as illustrated in the following example.

Example 8. Find $\mathcal{L}^{-1}\left(\dfrac{2s}{s(s^2+1)^2(s^2+4s+2)}\right)$ using partial fractions.

Solution. The partial fractions decomposition is

$$\frac{2s}{s(s^2+1)^2(s^2+4s+6)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2} + \frac{Fs+G}{s^2+4s+6}$$

Note the repeated factor $(s^2+1)^2$ lead to two partial fraction terms.

We won't compute the coefficients –you can do this by going through the algebra. Instead, we'll focus on finding the Laplace inverse. Our table contains the terms with repeated quadratic factors.

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{Ds}{(s^2+1)^2}\right) &= D\frac{t}{2}\sin(t) \\
\mathcal{L}^{-1}\left(\frac{E}{(s^2+1)^2}\right) &= E\frac{1}{2}(\sin(t) - t\cos(t)).
\end{aligned}
$$

For the terms with denominator s^2+s+2 we need to complete the square. Notice that we make sure to also shift the s-term in the numerator.

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{Fs+G}{s^2+4s+6}\right) &= \mathcal{L}^{-1}\left(\frac{F(s+2)+G-2F}{(s+2)^2+2}\right) \\
&= Fe^{-2t}\cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}(G-2F)e^{-2t}\sin(\sqrt{2}\,t).
\end{aligned}
$$

Putting this together, the answer is

$$\mathcal{L}^{-1}\left(\frac{2s}{s(s^2+1)^2(s^2+4s+6)}\right) = A + B\cos(t) + C\sin(t)$$

$$+ D\frac{t}{2}\sin(t) + E\frac{1}{2}(\sin(t) - t\cos(t))$$

$$+ Fe^{-2t}\cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}(G - 2F)e^{-2t}\sin(\sqrt{2}\,t).$$

7. Complex Factors

We can allow complex roots. In this case all quadratic terms factor into linear terms.

Example 9. Decompose $s/(s^2 + \omega^2)$ using complex partial fractions and use it to show $\mathcal{L}^{-1}(s/(s^2 + \omega^2)) = \cos(\omega t)$.

Solution.

$$\frac{s}{s^2 + \omega^2} = \frac{s}{(s - i\omega)(s + i\omega)} = \frac{A}{s - i\omega} + \frac{B}{s + i\omega}.$$

Multiplying through by the denominator gives $s = A(s + i\omega) + B(s - i\omega)$.
Plug in $s = i\omega \Rightarrow A = 1/2$.
Plug in $s = -i\omega \Rightarrow B = 1/2$.
From the table:

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + \omega^2}\right) = \mathcal{L}^{-1}\left(\frac{A}{s - i\omega} + \frac{B}{s + i\omega}\right) = Ae^{i\omega} + Be^{-i\omega} = \frac{1}{2}(e^{i\omega} + e^{-i\omega}) = \cos(\omega t).$$

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18.03SC Differential Equations
Fall 2011

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