Part II Problems and Solutions

Problem 1: *[Direction fields, isoclines] In this problem you will study solutions of the differential equation*

$$\frac{dy}{dx} = y^2 - x.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) *Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -1$, $m = 0$, $m = 1$, and $m = 2$. On this same graph, sketch, as best you can, a couple of solutions, using just the information given by these four isoclines.*

Having done this, you will continue to investigate this equation using one of the Mathlets. So invoke `http://math.mit.edu/mathlets/mathlets` *in a web browser and select* `Isoclines` *from the menu. (To run the applet from this window, click the little black box with a white triangle inside.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.*

Select from the pull-down menu our differential equation $y' = y^2 - x$. Move the m slider to $m = -2$ and release it; the $m = -2$ isocline is drawn. Do the same for $m = 0$, $m = 1$, and $m = 2$. Compare with your sketches. Then depress the mousekey over the graphing window and drag it around; you see a variety of solutions. How do they compare with what you drew earlier?

(b) *A separatrix is a curve such that above it solutions behave (as x increases) in one way, while below it solutions behave (as x increases) in quite a different way. There is a separatrix for this equation such that solutions above it grow without bound (as x increases) while solutions below it eventually decrease (as x increases). Use the applet to find its graph, and submit a sketch of your result.*
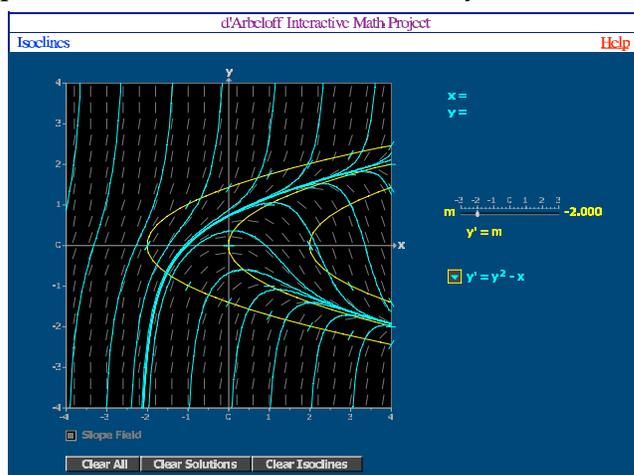
(c) *Suppose $y(x)$ is a solution to this differential equation whose graph is tangent to the $m = -1$ isocline: it touches the $m = -1$ isocline at a point (a, b), and the two curves have the same slope at that point. Find this point on the applet, and then calcluate the values of a and b.*

(d) *Now suppose that $y(x)$ is a solution to the equation for which $y(a) < b$, where (a, b) is the point you found in* **(c)**. *What happens to it as $x \to \infty$? I claim that its graph is asymptotic to the graph of $f(x) = -\sqrt{x}$. Explain why this is so. For large x, is $y(x) > f(x)$, $y(x) < f(x)$, or does the answer depend on the value of $y(a)$?*

The following observations will be useful in justifying your claims. Please explain as clearly as you can why each is true.
(i) The graph of $y(x)$ can't cross the $m = -1$ isocline at a point (x, y) with $x > a$.
(ii) If $c > a$ and $y(c)$ lies above the nullcline, then the graph of $y(x)$ continues to lie above the nullcline for all $x > c$.

(iii) If $c > a$ and $y(c)$ lies below the nullcline, then the graph of $y(x)$ will cross the nullcline for some $x > c$.

(e) *Suppose a solution $y(x)$ has a critical point at (c, d)—that is, $y'(c) = 0$ and $y(c) = d$. What can you say about the relationship between c and d? The applet can be very helpful here, but verify your answer.*

(f) *It appears from the applet that all critical points are local maxima. Is that true?*

Solution: **(a)** (The picture lacks the $m = 1$ isocline, sorry.)

(b) The separatrix is the elbow curve that other integral curves snuggle up to to the left and separate from as things move to the right. [Here are some facts about solutions to this differential equation, some of which are contained in later parts of this problem: Every solution has a "starting time" (if I think of x as time) before which it does not exist—every integral curve is asymptotic to a vertical straight line to its left. Every solution above the separatrix has an "ending time" after which it does not exist—it is asymptotic to a different vertical straight line to its right. Every solution below the separatrix survives forever, and becomes asymptotic (from above) to the graph of $y = -\sqrt{x}$. These are also exactly the solutions with a critical point—each has one local maximum. The separatrix itself is a solution, a very special one: it is the only solution which lives forever but has no maximum and is not asymptotic to $y = -\sqrt{x}$. Instead it is asymptotic to $y = \sqrt{x}$.]

(c) The point of tangency occurs where the tangent line to the $m = -1$ isocline has slope

-1. The $m = -1$ isocline is the locus of $x = y^2 + 1$. Differentiate implicitly (using the chain rule) $1 = 2yy'$, so if $y' = -1$ then $y = -1/2$. The equation then gives $x = 5/4$, so $(a, b) = (5/4, -1/2)$.

(d) (i) Follow the solution from the $x = a$ line. Suppose that it does cross the $m = -1$ isocline, and suppose that the first crossing is at a point (x, y). Since it started below the $m = -1$ isocline, it must be crossing it from below. This means that the slope of the solution must be greater than the slope of the isocline at the crossing point. The slope of the isocline at the crossing point is greater than -1, so the slope of the solution at the crossing point must be greater than -1 too. But when a solution crosses the -1 isocline, it must cross it with slope exactly -1. So it can't cross it at all.

(ii) Suppose that in fact the graph of $y(x)$ does cross the nullcline. The slope of the nullcline is negative; the first crossing must be from above; so at the crossing the slope of the solution must be even more negative; but when it crosses the nullcline it must have slope zero.

(iii) Finally I claim that if $(a, y(a))$ is below the nullcline, then the integral curve must eventually cross the nullcline. Until it does, it is in the region where the direction field has positive slopes. So it is increasing. But the lower branch of the nullcline is is falling; so the two curves must eventually intersect.

One can also argue that if the solution never crosses the nullcline then, after rising at the start it must eventually start to fall, since the nullcline does. So it must reach a maximum somewhere in between. But all maxima of solutions occur along the nullcline.

(e) Critical points of solutions of $y' = y^2 - x$ occur when $y' = 0$, that is, along the null-cline: so if there is one at (c, d) then $c = d^2$.

(f) Yes. Here are two arguments: All solutions are increasing for small values of x. From what we can see, the solutions with critical points are the ones which become asymototic to the graph of $y = -\sqrt{x}$, which decreases. Also they seem to have just one critical point, which must therefore be a maximum. More complete argument: Compute (as in **(c)**) $y'' = 2yy' - 1$. So at a critical point, where $y' = 0$, we see that $y'' = -1$. This means maximum.

18.03SC Differential Equations

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