18.034 Honors Differential Equations
Spring 2009

LECTURE 22. CONVOLUTION

Motivation: buildup of a pollutant in a lake. Let's say we have a lake and a pollutant is being dumped into it at the variable rate $f(t)$. The pollutant degrades over time exponentially. If the lake begins at $t = 0$ with no pollutant, how much is in the lake at time $t > 0$?

The small drip of pollutant added to the lake between t_1 and $t_1+\Delta t$, where Δt small, is $f(t_1)\Delta t$. Later $t > t_1$, the drip reduces to $e^{-a(t-t_1)}f(t_1)\Delta t$, where $a > 0$ is the decay constant. Adding them up, starting at the initial time $t_1 = 0$, we obtain that the amount is

$$\int_0^t e^{-a(t-t_1)} f(t_1)\, dt_1. \tag{22.1}$$

Integral of this kind is called a *convolution*.

We can solve this problem by setting up a differential equation. Let $y(t)$ be the amount of pollutant in the lake at time t. Then, the amount of the chemical in the lake at time $t + \Delta t$ is the amount at time t, minus the fraction that decayed plus the amount newly added:

$$y(t + \Delta t) = y(t) - ay(t)\Delta t + f(t)\Delta t.$$

Taking the limit as $\Delta t \to 0$ we obtain

$$y' + ay = f(t), \qquad y(0) = 0.$$

It is straightforward that (22.1) gives the solution of the above initial value problem.

The convolution integral. The convolution of f and g is defined as

$$(f * g)(t) = \int_0^t f(t_1)g(t - t_1)\, dt_1. \tag{22.2}$$

It gives the response at the present time t as a weighted superposition over the input at times $t_1 < t$. The weight $g(t - t_1)$ characterizes the system and $f(t_1)$ characterizes the history of the input. To ensure the existence of the integral, in what follows, we assume that $f, g \in A$.

Example 22.1. Let $f(t) = e^{B_1 t}$ and $g(t) = e^{B_2 t}$, where $B_1 \neq B_2$ are constants. Then,

$$(f * g)(t) = \int_0^t e^{B_1 t} e^{B_2(t-t_1)}\, dt_1 = \frac{e^{B_1 t} - e^{B_2 t}}{B_1 - B_2}.$$

Since $\mathcal{L}e^{at} = 1/(s - a)$ for any constant a, we have

$$\mathcal{L}(f * g) = \frac{1}{B_1 = B_2}\left(\frac{1}{s - B_1} - \frac{1}{s - B_2}\right) = \frac{1}{s - B_1}\frac{1}{s - B_2} = (\mathcal{L}f)(\mathcal{L}g).$$

It is not a coincidence, but rather, it is a property of convolution, as discussed below.

The convolution operator acts like ordinary multiplication in that, if f, g, h are admissible then

(i) (distibutive) $f * (g + h) = f * g + f * h$,
(ii) (commutative) $f * g = g * f$,
(iii) (associative) $f * (g * h) = (f * g) * h$.

However, the convolution operator differs from the multiplication operator. For example, $f*1 \neq f$, $f*f \neq f^2$ in general.

Exercise. Show that $t^2 * 1 = t^3/3$ and $\cos t * \cos t = \frac{1}{2}(t\cos t + \sin t)$.

Nevertheless, convolution in the t-domain does corresponds to multiplication in the s-domain.

Theorem 22.2 (Convolution Theorem). *If $f, g \in E$, then $f * g \in E$ and $\mathcal{L}(f*g) = (\mathcal{L}f)(\mathcal{L}g)$.*

Proof. If $f, g \in E$, then $f * g$ is continuous for all $t \in [0, \infty)$. Since $|f(t)| \leqslant A_1 e^{B_1 t}$ and $|g(t)| \leqslant A_2 e^{B_2 t}$, then

$$|(f*g)(t)| \leqslant \int_0^t A_1 e^{B_1 t} A_2 e^{B_2(t-t_1)}\, dt_1 = A_1 A_2 \frac{e^{B_1 t} - e^{B_2 t}}{B_1 - B_2}.$$

Therefore, $f * g \in E$. Better yet, $|f| * |g| \in E$. In other words, $\mathcal{L}(f*g)$ converges absolutely for large s.

For simplicity, let $f(t) = 0$ and $g(t) = 0$ for $t < 0$, so that

$$\mathcal{L}f = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt, \quad \mathcal{L}g = \int_{\infty}^{\infty} e^{-st} g(t)\, dt, \quad \text{and} \quad (f*g)(t) = \int_{-\infty}^{\infty} f(t_1) g(t-t_1)\, dt_1.$$

Indeed, the lower limit of the integral of $f * g$ can be replaced by $-\infty$ since $f(t_1) = 0$ for $t_1 < 0$ and the upper limit can be replaced by ∞ since $g(t - t_1) = 0$ for $t - t_1 < 0$. Then,

$$\begin{aligned}
\mathcal{L}(f*g)(s) &= \int_{-\infty}^{\infty} e^{-st} \int_{-\infty}^{\infty} f(t_1) g(t-t_1)\, dt_1\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-st} g(t-t_1)\, dt \right) f(t_1)\, dt_1 \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-s(t_1+t_2)} g(t_2)\, dt_2 \right) f(t_1)\, dt_1 \\
&= \int_{-\infty}^{\infty} e^{-st_1} \mathcal{L}g(s) f(t_1)\, dt_1 = (\mathcal{L}f)(\mathcal{L}g).
\end{aligned}$$

□

Convolution allows an easy passage from the s-domain to the t-domain and leads to explicit solutions for a general inhomogeneous term $f(t)$.

Example 22.3. Solve the initial value problem

$$y'' + \omega^2 y = \omega^2 f(t), \qquad y(0) = y'(0) = 0,$$

where ω^2 is constant and $f \in E$.

SOLUTION. Taking the transform,

$$(s^2 + \omega^2)\mathcal{L}y = \omega^2 \mathcal{L}f, \qquad \mathcal{L}y = \mathcal{L}f \frac{\omega^2}{s^2 + \omega^2}.$$

Since $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}$, it follows that $\mathcal{L}y = (\mathcal{L}f)(\mathcal{L}\omega \sin \omega t)$. By the convolution theorem and by uniqueness, then

$$y(t) = f * (\omega \sin \omega t) = \omega \int_0^t f(t_1) \sin \omega(t - t_1)\, dt_1.$$

Note that we have a formula for the rest solution corresponding to the arbitrary function f.

The tautochrone. Starting from the rest state, suppose that a particle of mass m slides down a frictionless curve under gravity as shown in the figure below.

The aim is to determine the shape of the curve so that the time of descent is independent of the starting point. A curve of this kind is called a *tautochrone*. It comes from two Greek words, meaning "same" and "time." The problem was solved by the Dutch mathematician Christian Huygens in 1673 as part of his theory of pendulum clocks.

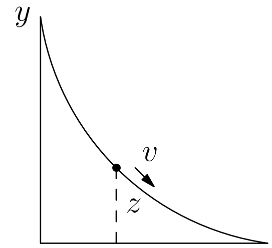

Figure 22.1. The tautochrone.

Let y be the height at which the particle starts and let v be the speed of the particle at the height z. The change in the kinetic energy $(1/2)mv^2$ at the height z is the change in the potential energy, which leads to

$$\frac{1}{2}mv^2 = mg(y-z), \qquad v = \sqrt{2g}\sqrt{y-z}, \tag{22.3}$$

where g is the gravitational acceleration. Let $\sigma = \sigma(y)$ be the arc from the rest state to the lowest point. The time of descent is

$$\int_0^{\sigma(y)} \frac{d\sigma}{v} = \int_0^y \frac{1}{v}\frac{d\sigma}{dz}\,dz \equiv \int_0^y \frac{1}{v}\phi(z)\,dz.$$

Here, $\phi(y) = d\sigma/dy$ so that $\phi(z) = d\sigma/dy|_{y=z}$. Since the time of decent is constant, in view of (22.3), the problem reduces to

$$\int_0^y \phi(z)(y-z)^{-1/2}\,dz = c_1,$$

where c_1 is constant. The left side is the convolution of ϕ and $y^{-1/2}$. Taking the transform, then

$$\mathcal{L}(\phi * y^{-1/2}) = (\mathcal{L}\phi)(\mathcal{L}y^{-1/2}) = \mathcal{L}c_1 = \frac{c_1}{s}.$$

The first equality uses the convolution theorem.

Since* $\mathcal{L}[t^{-1/2}] = \sqrt{\pi}s^{-1/2}$, we have

$$\mathcal{L}\phi = c_2 s^{-1/2}, \qquad \phi(y) = cy^{-1/2},$$

where c, c_2 are constants. Since $\phi(y) = d\sigma/dy$, the equation of the curve reduces to

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{c^2}{y} \quad \text{or} \quad dx = \sqrt{\frac{c^2}{y} - 1}\,dy.$$

Exercise. By parameterizing $y = c^2 \sin^2\theta$, show that the curve is a cycloid

$$x = \frac{c^2}{2}(2\theta + \sin 2\theta), \qquad y = \frac{c^2}{2}(1 - \cos 2\theta).$$

When a particle slides down a frictionless curve, a related question is to find a path which minimizes the time of descent. The minimizing curve is called a "brachistochrone". It is interesting that the solution is again a cycloid. (not a straight line!)

*See the pset question on the Gamma function.