18.034 Honors Differential Equations
Spring 2009

LECTURE 2. FUNDAMENTAL PRINCIPLES

Linearity. A (differential) operator L is said *linear* if

(2.1) $$L(c_1 u + c_2 v) = c_1 L u + c_2 L v$$

for all scalars c_1, c_2 and for all members u, v in the domain of L. For example, the differential operator, defined by $Ly = y' + p(x)y$ is linear, where $p(x)$ is defined on an open interval I and the domain of L consists of differentiable functions on I.

The importance of linear operators lies in the following.

Principle of Superposition. If L is a linear operator, then $Lu = f$ and $Lv = g$ implies

$$L(c_1 u + c_2 v) = c_1 f + c_2 g.$$

The proof is immediate and it is left as an exercise. The principle of superposition has application in the study of linear differential equations.

Example 2.1. Let

$$Ly = y'' + y.$$

It is obvious that $L(\sin x) = 0$ and $L(\cos x) = 0$. Then, the principle of superposition show that $y(x) = c_1 \sin x + c_2 \cos x$ satisfies $Ly = 0$ for any constants c_1, c_2. Thus, we obtain a two-parameter family of solutions of $y'' + y = 0$.

Example 2.2. Let

$$Ly = y' - 2y.$$

By inspection,

$$L(1) = -2, \qquad L(e^{5x}) = 3e^{5x}, \qquad L(e^{2x}) = 0.$$

then, the principle of superposition show that $y = -4 + 2e^{5x} + ce^{2x}$ is a solution of $y' - 2y = 8 + 6e^{5x}$ for any constant c.

Exercise. (Principle of the Complementary Solution.) Let L be a linear operator. If u is a particular solution of $Ly = f$ and if v is a solution of $Ly = 0$, then show that $y = u + v$ satisfies $Ly = f$.

Existence and uniqueness. When a problem from physics or engineering is formulated into a set of differential equations, demanded are that a solution should exist and that it should be unique. They are the subjects of *existence* and *uniqueness*, respectively. The difference between two issues is: an existence result asserts that there is *at least* one solution while a uniqueness result asserts that there is *at most* one solution.

While existence theory is not slighted in this course, many differential equations are easy to solve on computers with approximate solutions accurate to (say) 3-5 decimal digits. (For special classes of differential equations which one can solve explicitly, existence issue is settled then and there.) On the other hand, uniqueness is a theoretical issue and it can be studied only by means of analytical proofs.

To put this in perspective, let us study a physical system described by

$$y'' + y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

It is easily seen that $y(x) = \sin x$ is a solution; the issue of existence is settled. This particular solution describes an oscillatory behavior, and it is tempted to say that the system describes an oscillatory motion. But, with the uniqueness issue in doubt, we cannot make such a conclusion.

Uniqueness indeed fails. Let us consider

$$y'' + y = 0, \qquad y(0) = 0 = y(\pi).$$

Again, $y(x) = \sin x$ is a solution, and the existence is a trivial matter. But, $y(x) = A\sin(kx)$ is a solution for any $A \in \mathbb{R}$ and $k \in \mathbb{Z}$. There are infinitely many solutions, therefore. This system consists of the simple second-order linear differential equation, same as one in above and two simple side conditions. In fact, the system describes small oscillations of a plucked string, and it is a perfectly good physical system.

We give another example of failure of uniqueness.

Example 2.3. Let us consider the differential equation

(2.2)
$$y' = 3y^{2/3}.$$

It is straightforward that $y(x) = (x - c)^3$ is a solution of the differential equation for any c. But, there are other solutions. Indeed, $y(x) = 0$ is also a solution of (2.2). Moreover,

$$y(x) = \begin{cases} (x - a)^3 & \text{for} \quad x < a, \\ 0 & \text{for} \quad a \leqslant x < b \\ (x - b)^3 & \text{for} \quad x \geqslant b \end{cases}$$

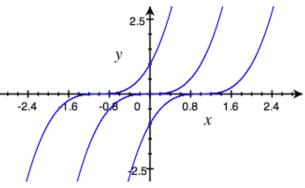is a solution of (2.2) for any $a < b$, sketched below.

Figure 2.1. Solutions of (2.2).

Hence, (2.2) has a *two-parameter* family of solutions, depending on a and b.

Failure of uniqueness is not always a bad news. A point where a presumably unique solution branches into multiple solutions is called *bifurcation point*. At such points, the system experiences a sudden change in qualitative behavior. Bifurcation is a special kind of failure of uniqueness and it has significance in applications. e.g. consider

(2.3)
$$y' = (a - c)y - by^3, \qquad b > 0.$$

Here, we view b and c as fixed and a as a parameter. If $a < c$, then $y = 0$ is the only stationary solution of the differential equation. But, if $a > c$, then in addition to $y = 0$, there are two more stationary solutions $y = \pm\sqrt{(a - c)/b}$. In other words, at $a = c$ there bifurcate nontrivial stationary solutions from a line of trivial stationary solution. In this case, $a = c$ is a bifurcation point.

We now present a basic uniqueness theorem.

Theorem 2.4 (A Uniqueness Theorem). *If $f(y)$ is a continuously differentiable and bounded in a rectangular region R in the (x, y)-plane, and if, in addition, f' is bounded in R, then for any point $(x_0, y_0) \in R$, the differential equation*

$$y' = f(y)$$

has at most one solution satisfying $y(x_0) = y_0$.

The proof is presented later with more generality.

Remark. If $f(y) = 3y^{2/3}$ then $f'(y) = 2y^{-1/3}$ is unbounded on a region containing zero, and it does not violate the theorem.

Exercise. Show that the solution curves of (2.2) and $y(0) = 0$ fill up entirely the region between the curve $y = x^3$ and the x-axis. That is, for any point (x_0, y_0) in this region, there is a solution of (2.2) satisfying $y(x_0) = y_0$.

Qualitative behavior. The equation $dy/dx = f(y)$ gives the slope of a solution $y = \phi(x)$ at any point of the curve, even before solving the equation, and it gives a qualitative behavior of the system.

Let us consider the differential equation

(2.4) $$\frac{dy}{dx} = (y-1)(y-2)(y-4) \equiv f(y).$$
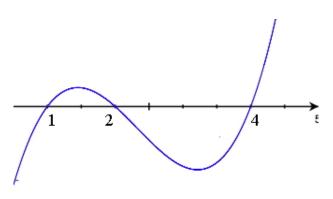
The graph of $f(y)$ is give below.

Figure 2.3. Graph of $f(y)$.

It is readily seen that (2.4) has the stationary solutions $y = 1, 2, 4$. At these stationary solutions, $f(y) = 0$ and hence $dy/dx = 0$. Therefore, the solution curves are horizontal.

Now, we study the behavior of nonstationary solutions of (2.4) in terms of the sign of $f(y)$. If $y < 1$, then $f(y)$ is negative, and the solutions are increasing functions. That means, solutions move away from $y = 1$. Similarly, if $1 < y < 2$ then $f(y)$ is positive, and solutions move away from $y = 1$ and move towards $y = 2$. If $2 < y < 4$ solutions move towards $y = 2$ and move away from $y = 4$. Finally, if $4 < y$ then solutions move away from $y = 4$. For these reasons, we say the stationary solution $y = 2$ is *stable* and $y = 1, 4$ are *unstable*. The behavior of solution curves is sketched in the figure below.
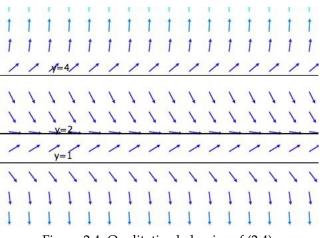
Figure 2.4. Qualitative behavior of (2.4).

Here, we rely on the intuitive definitions of stability/instability. The rigorous definitions of stability/instability will be given later.

Let us consider a more complicated example

$$\frac{dy}{dx} = (y+1)^3 y^4 (y-2)^7 (y-3)^5 (y-5)^8 \equiv g(y).$$

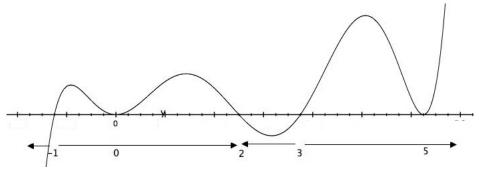The graph of $g(y)$ and its sign are indicated in the figure below.

Figure 2.5. Graph of $g(y)$ and its change of sign.

By the same argument that we used before we say the stationary solutions $y = -1, 3$ are unstable while $y = 2$ is stable. We say the stationary solutions $y = 0, 5$ are *semi-stable*. For, on one side of $y = 0$ or $y = 5$, the adjacent solutions move towards the stationary solution, but on the other side solutions move away from it.