Recitation 13, March 18, 2010

Fourier Series: Introduction

1. What is the general solution to $\ddot{x} + \omega_n^2 x = 0$? [Quick!]

The characteristic polynomial is $p(s) = s^2 + \omega_n^2$, with roots $\pm i\omega_n$. The general complex solution is $ae^{i\omega_n t} + be^{-i\omega_n t}$, and the general real solution is $x = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$.

2. Discuss why (as long as $\omega \neq \pm\omega_n$)

$$\ddot{x} + \omega_n^2 x = a\cos(\omega t) \quad \text{has solution} \quad x_p = a\,\frac{\cos(\omega t)}{\omega_n^2 - \omega^2}$$

$$\ddot{x} + \omega_n^2 x = b\sin(\omega t) \quad \text{has solution} \quad x_p = b\,\frac{\sin(\omega t)}{\omega_n^2 - \omega^2}$$

Assume $\omega \neq \omega_n$, then the first equation is the real part of the equation $\ddot{z} + \omega_n^2 z = ae^{i\omega t}$. By the exponential response formula, $z = \frac{a}{\omega_n^2 - \omega^2} e^{i\omega t}$. Taking the real part yields solution $x_p = a\frac{\cos(\omega t)}{\omega_n^2 - \omega^2}$. Similarly, the second equation is the imaginary part of the equation $\ddot{z} + \omega_n^2 z = be^{i\omega t}$, which has a solution $z = \frac{b}{\omega_n^2 - \omega^2} e^{i\omega t}$. Taking the imaginary part yields $x_p = b\frac{\sin(\omega t)}{\omega_n^2 - \omega^2}$.

3. What about $\ddot{x} + \omega_n^2 x = \cos(\omega_n t)$? What is a particular solution? What is the general solution? Are there any solutions $x(t)$ such that $|x(t)| < 10^6$ for all t? Are there any periodic solutions?

Assume $\omega_n \neq 0$, otherwise the equation becomes trivial. Again we take the complex replacement of the equation and get $\ddot{z} + \omega_n^2 z = e^{i\omega_n t}$. Since $p(i\omega_n) = 0$, $p'(i\omega_n) = 2i\omega_n \neq 0$, then by RERF, the particular solution is $z_p(t) = \frac{te^{i\omega_n t}}{2i\omega_n}$. Taking the real part yields $x_p(t) = \frac{t\sin(\omega_n t)}{2\omega_n}$. The solution of the homogeneous equation is given by problem 1, so the general solution of this equation is $x(t) = \frac{t\sin(\omega_n t)}{2\omega_n} + c_1\cos(\omega_n t) + c_2 \sin(\omega_n t)$.

Due to the factor t in x_p, any solution $x(t)$ will exit the range $(-10^6, 10^6)$ when t is large enough. To see this, consider $t_k = \frac{2k+1/2}{\omega_n}\pi$ for $k \in \mathbb{N}$. Then $\sin(\omega_n t_k) = \sin(2k\pi + \pi/2) = 1$ and $\cos(\omega_n t_k) = \cos(2k\pi + \pi/2) = 0$, so $x(t_k) = \frac{2k+1/2}{2\omega_n^2}\pi + c_2$. Therefore for any c_2, we can always choose sufficiently large integer k , such that $x(t_k) = \frac{2k+1/2}{2\omega_n^2}\pi + c_2 > 10^6$. So there will be NO solution $x(t)$ such that $|x(t)| < 10^6$ for all t.

$c_1\cos(\omega_n t) + c_2\sin(\omega_n t)$ is a periodic function with period $2\pi/\omega_n$, but $x_p(t) = \frac{t\sin(\omega_n t)}{2\omega_n}$ is not periodic since the magnitude increases as t increases. So the general solution $x(t) = x_p(t) + c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$ is not periodic. There is NO periodic solution.

4. On the same set of axes, sketch graphs of $\sin t$, $\sin(2t)$. Then sketch the graph of $f(t) = \sin t + \sin(2t)$. Some pointers: $f(t)$ is easy to evaluate when one of the terms is zero. What is the derivative at points where both terms are zero? This information should be enough to let you make a rough sketch. What are the periods of these three functions?

$\sin t$ has period 2π, and $\sin(2t)$ has period π. Both $\sin t$ and $\sin(2t)$ will vanish at $t = k\pi$ for $k \in \mathbb{Z}$. $f'(t) = \cos t + 2\cos(2t)$, so at those points, $f'(k\pi) = (-1)^k + 2 = \begin{cases} 1, & k\, odd, \\ 3, & k\, even. \end{cases}$ $f(t)$ is a linear combination of $\sin t$ and $\sin(2t)$, so its period is the least common multiple of the periods of $\sin t$ and $\sin(2t)$, i.e., 2π.

5. For what values of ω_n is there a periodic solution to the equation

$$\ddot{x} + \omega_n^2 x = b_1 \sin t + b_2 \sin(2t)$$

(where b_1 and b_2 are nonzero)? Name one if it exists.

Consider the complex replacement of the equation $\ddot{z} + \omega_n^2 z = b_1 e^{it} + b_2 e^{2it}$. The characteristic polynomial is $p(s) = s^2 + \omega_n^2$. By linearity, a solution to this equation is given by the sum of a solution to $\ddot{z} + \omega_n^2 z = b_1 e^{it}$ and a solution to $\ddot{z} + \omega_n^2 z = b_2 e^{2it}$. When $\omega_n \neq 1$, $\ddot{z} + \omega_n^2 z = b_1 e^{it}$ has the solution $b_1 e^{it} / (\omega_n^2 - 1)$; when $\omega_n \neq 2$, $\ddot{z} + \omega_n^2 z = b_2 e^{2it}$ has the solution $b_2 e^{2it} / (\omega_n^2 - 4)$. So if both $\omega_n \neq 1$ and $\omega_n \neq 2$ are satisfied, a complex solution of the original equation will be $\frac{b_1 e^{it}}{\omega_n^2 - 1} + \frac{b_2 e^{2it}}{\omega_n^2 - 4}$. Taking the imaginary part yields $x_p = \frac{b_1 \sin t}{\omega_n^2 - 1} + \frac{b_2 \sin(2t)}{\omega_n^2 - 4}$, which is a periodic solution with period 2π.

6. (very tricky) For what values of ω is $\sin t + \sin(\omega t)$ periodic? And the periods?

When $\omega = 0$, the function is simply $\sin t$ which is periodic with period 2π. Now assume $\omega \neq 0$, then $\sin(\omega t)$ is periodic with period $2\pi/\omega$. Since the period of $\sin t$ is 2π, then if $\sin t + \sin(\omega t)$ is about to be periodic, its period must be $2k\pi$ for some positive integer $k \in \mathbb{N}$, which also has to be an integer multiple of the period of $\sin(\omega t)$. Therefore if $\sin t + \sin(\omega t)$ is periodic, then there exist positive $k \in \mathbb{N}$ and positive $n \in \mathbb{N}$, such that $2k\pi = n\frac{2\pi}{\omega}$, i.e., $\omega = n/k$. On the other hand, for any choice of a pair of positive integers k and n, set $\omega = n/k$, then $\sin t + \sin(\omega t)$ will be periodic with period $\frac{2k\pi}{(k, n)}$, where (k, n) denotes the greatest common divisor of k and n.