Recitation 3, February 9, 2010

Euler's method; Linear models

Solution suggestions

1. Use Euler's method to estimate the value at $x = 1.5$ of the solution of $y' = y^2 - x^2 = F(x, y)$ at $y(0) = -1$. Use $h = 0.5$. Recall the notation $x_0 = 0$, $y_0 = -1$, $x_{n+1} = x_h + h$, $y_{n+1} = y_n + m_n h$, $m_n = F(x_n, y_n)$. Make a table with columns n, x_n, y_n, m_n, $m_n h$. Draw the Euler polygon.

n	x_n	y_n	m_n	$m_n h$
0	0	-1	1	0.5
1	0.5	-0.5	0	0
2	1.0	-0.5	-0.75	-0.375
3	1.5	-0.875		

2. Is the estimate from **1.** likely to be too large or too small?

It is likely to be too large. One way of seeing this intuitively is to take the derivative of the equation, and note that y" is negative on the points in question 1. Since any tangent to a concave function lies above the function, this suggests the Euler estimate is greater than the actual solution. To prove it, it suffices to show that the slope field always crosses the Euler approximate solution in a downward direction, i.e., y' is at most the slope of the line segment connecting two successive points. You can do this by parametrizing the line segments, feeding the points into the differential equation to get y' as a function of x, then comparing to the slopes of the line segments.

First: $y = x - 1$ with slope 1, so $y' = y^2 - x^2 = (x - 1)^2 - x^2 = -2x + 1$. When $x \in [0, 0.5]$, $-2x + 1 \leq 1$.

Second: $y = -0.5$ with slope zero, so $y' = y^2 - x^2 = 0.25 - x^2$. When $x \in [0.5, 1]$, this is nonpositive.

Third: $y = -0.75x + 0.25$ with slope -0.75, so $y' = y^2 - x^2 = -0.4375x^2 - 0.375x + 0.0625$. We would like to compare this with -0.75 in the interval $[1, 1.5]$. At $x = 1$, we have equality, so it suffices to show that the first derivative of y' on the segment, or $y" = -0.875x - 0.375$, is nonpositive for $x \in [1, 1.5]$, and it is.

3. Here's a "mixing problem." A tank holds V liters of salt water. Suppose that a saline solution with concentration of c gm/liter is added at the rate of r liters/minute. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of r liters/minute. Write down the differential equation for the *amount* of salt in the tank. [Not

the concentration!] Use the notation $x(t)$ or the number of grams of salt in the tank at time t. Check the units in your equation! Write it in standard linear form.

The concentration of salt at any given time is $x(t)/V$ gm/liter, so for small Δt, we lose $rx(t)\Delta t/V$ gm from the exit pipe, and we gain $rc\Delta t$ gm from the input pipe. The equation is $x'(t) = rc - \frac{rx(t)}{V}$, and in standard linear form, it is

$$x' + \frac{r}{V}x - rc = 0.$$

4. Now assume that c and r are constant; in fact, assume that $V = 1$ and $r = 2$. Solve this equation, under the assumption that $x(0) = 0$. What is the limiting amount of salt in the tank? Does your result jibe with simple logic? When will the tank contain half that amount?

Our new equation is $x' + 2x - 2c = 0$. We substitute: $u = x - c$, so $u' = x' = -2x+2c = -2u$. Solving gives us $u = Ce^{-2t}$, and $x = u+c = Ce^{-2t}+c$. With the initial value $x(0) = 0$, we have $Ce^0+c = 0$, or $C = -c$, so $x(t) = -ce^{-2t}+c$.

The limiting amount of salt is $c = cV$ gm, and that is what we should expect, since the concentration should approach that of the inflow. To get half the salt, we solve $-ce^{-2t} + c = c/2$, which yields $t = \frac{\ln 2}{2}$.

5. Now suppose that the out-flow from this tank leads into another tank, also of volume 1, and that at time $t = 1$ the water in it has no salt in it. Again there is a mixer and an outflow. Write down a differential equation for the amount of salt in this second tank, as a function of time.

Assume the amount of salt in the second tank at moment t is given by $y(t)$. We use the equation for the first tank, but the concentration of the inflow has become $x(t)$. The differential equation for $y(t)$ is

$$y' + ry - rx(t) = 0.$$

6. Draw a picture of the circuit with a voltage source, a resistor, and a capacitor, in series. Denote by $I(t)$ the current (where the positive direction is, say, clockwise) in the circuit and by $V(t)$ the voltage increase across the voltage source, at time t. Denote by R the resistance of the resistor and C the capacitance of the capacitor (in units which we will not specify)—both positive numbers. Then

$$R\dot{I} + \frac{1}{C}I = \dot{V}$$

Suppose that V is constant, $V(t) = V_0$. Solve for $I(t)$, with initial condition $I(0)$.

It is common to write the solution in the form $ce^{-t/\tau}$. Calculate c and τ. Note that τ is measured in the same units as t (in order for the exponent to be dimensionless). It is called the *characteristic time* for the system. What is $I(t + \tau)$ in terms of $I(t)$?

Turn to EP page 173 for a picture of an electrical circuit, but the circuit in question 6 is without an inductor.

When V is constant, the equation becomes $R\dot{I} + \frac{1}{C}I = 0$, which is separable. Solving gives us

$$I\left(t\right) = I\left(0\right)e^{-\frac{t}{CR}},$$

and $c = I\left(0\right)$, $\tau = CR$. $I\left(t+\tau\right) = ce^{-\frac{t}{\tau}}e^{-1} = I\left(t\right)/e$.

18.03 Differential Equations
Spring 2010