Problems: Harmonic Functions and Averages

A function u is called *harmonic* if $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. In this problem we will see that the average value of a harmonic function over any sphere is exactly its value at the center of the sphere.

For simplicity, we'll take the center to be the origin and show the average is $u(0,0,0)$.

Let u be a harmonic function and S_R the sphere of radius R centered at the origin. The average value of u over S is given by $A = \frac{1}{4\pi R^2} \iint_S u(x,y,z)\, dS$.

1. Write this integral explicitly using spherical coordinates.

<u>**Answer:**</u>

$$\begin{aligned} A &= \frac{1}{4\pi R^2} \int_0^{2\pi} \int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\, R^2 \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\, \sin\phi\, d\phi\, d\theta. \end{aligned}$$

2. Differentiate A with respect to R

<u>**Answer:**</u> $\dfrac{dA}{dR} = \dfrac{1}{4\pi} \displaystyle\int_0^{2\pi} \int_0^{\pi} (u_x \sin\phi\cos\theta + u_y \sin\phi\sin\theta + u_z\cos\phi)\, \sin\phi\, d\phi\, d\theta.$

3. Rewrite the formula in part (2) in terms of $\nabla u \cdot \mathbf{n}$.

<u>**Answer:**</u> On S we have $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{R} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ and $dS = R^2 \sin\phi\, d\phi\, d\theta$

$$\Rightarrow \frac{dA}{dR} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \langle u_x, u_y, u_z\rangle \cdot \mathbf{n}\, \frac{dS}{R^2} = \frac{1}{4\pi R^2} \iint_{S_R} \nabla u \cdot \mathbf{n}\, dS.$$

4. Use the divergence theorem to show $\frac{dA}{dR} = 0$ and conclude the average $A = u(0,0,0)$.

<u>**Answer:**</u> Let D be the solid ball of radius R. Applying the divergence theorem to part (3) we get

$$\frac{dA}{dR} = \frac{1}{4\pi R^2} \iiint_D \nabla \cdot \nabla u\, dV = \frac{1}{4\pi R^2} \iiint_D \nabla^2 u\, dV = 0.$$

For R near 0 the average is approximately $u(0,0,0)$.

Since the derivative is 0 the average is the same for any radius and we can let R go to 0 to conclude $A = u(0,0,0)$.

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