Problems: Triple Integrals

1. Set up, but do not evaluate, an integral to find the volume of the region below the plane $z = y$ and above the paraboloid $z = x^2 + y^2$.

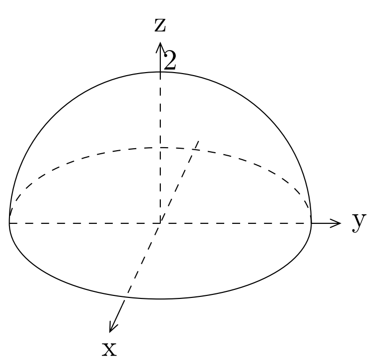
Answer: Draw a picture. The plane $z = y$ slices off an thin oblong from the side of the paraboloid. We'll compute the volume of this oblong by integrating vertical strips in the z direction over a region in the xy-plane.

To describe the planar region below the volume, we study the curve of intersection of the plane and the paraboloid: $y = x^2 + y^2$. Completing the square gives us $\frac{1}{4} = x^2 + \left(y - \frac{1}{2}\right)^2$. This is the equation of a circle with radius $1/2$ about the center $(0, 1/2)$. (We might also discover this by solving to get $x = \pm\sqrt{y - y^2}$ and using a computer graphing utility.)

The most natural set of limits seems to be:
Inner: z from $x^2 + y^2$ to y.
Middle: x from $-\sqrt{y - y^2}$ to $\sqrt{y - y^2}$.
Outer: y from 0 to 1.

Thus, Volume $= \displaystyle\int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \int_{x^2+y^2}^{y} 1\, dz\, dx\, dy.$

2. Use cylindrical coordinates to find the center of mass of the hemisphere shown. (Assume $\delta = 1$.)



$x^2 + y^2 + z^2 = 4$

Answer: By symmetry it's clear $x_{cm} = 0$ and $y_{cm} = 0$.
$z_{cm} = \dfrac{1}{M} \displaystyle\iiint_D z\, dm = \dfrac{1}{M} \iiint_D z\, \delta\, dV.$

Clearly R is a disc of radius 2 and $M = \dfrac{16}{3}\pi$.

Limits: inner z: from 0 to $\sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$.
middle r: from 0 to 2.
outer θ: from 0 to 2π.

$\Rightarrow\ z_{cm} = \dfrac{3}{16\pi} \displaystyle\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} zr\, dz\, dr\, d\theta.$

Inner: $\dfrac{3}{16\pi}\dfrac{z^2r}{2}\Bigg|_0^{\sqrt{4-r^2}} = \dfrac{3}{16\pi}\dfrac{4-r^2}{2}\cdot r = \dfrac{3}{16\pi}\dfrac{4r-r^3}{2}$.

Middle: $\dfrac{3}{16\pi}\left[r^2 - \dfrac{r^4}{8}\right|_0^2 = \dfrac{3}{8\pi}$.

Outer: $\dfrac{3}{8\pi}2\pi = \dfrac{3}{4} \Rightarrow z_{cm} = \dfrac{3}{4}$.

It makes sense that the center of mass would lie between $(0, 0, 0)$ and $(0, 0, 1)$.