Green's Theorem and Conservative Fields

We can use Green's theorem to prove the following theorem.

Theorem

Suppose $\mathbf{F} = \langle M, N \rangle$ is a vector field which is defined and with continuous partial derivatives for all (x, y). Then

$$\mathbf{F} \text{ is conservative } \Leftrightarrow N_x = M_y \text{ or } N_x - M_y = \operatorname{curl} \mathbf{F} = 0.$$

Proof

This is a consequence of Green's theorem. First, suppose $\mathbf{F}$ is conservative, i.e., its work integral is 0 along all simple closed curves. Then Green's theorem says

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R \operatorname{curl} \mathbf{F} \, dA.$$

The only way for the integral of $\operatorname{curl} \mathbf{F}$ to be 0 over all regions R is if $\operatorname{curl} \mathbf{F}$ itself is 0. This implies $N_x = M_y$ as claimed.

For the converse, assume $N_x = M_y$. Then, for any closed curve C surrounding a region R, Green's theorem says,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R N_x - M_y \, dA = 0.$$

Therefore, the work integral of $\mathbf{F}$ is 0 over any closed curve, which means $\mathbf{F}$ is conservative.

Be careful, the requirement that $\mathbf{F}$ is defined and differentiable everywhere is important. The problem following this note will give an example of a nonconservative field with $\operatorname{curl} \mathbf{F} = 0$.

Later we will learn how to handle fields that aren't defined everywhere.