V2.1 Gradient Fields and Exact Differentials

1. Criterion for gradient fields.

Let $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a two-dimensional vector field, where M and N are continuous functions. There are three equivalent ways of saying that $\mathbf{F}$ is conservative, i.e., a gradient field:

$$(1) \qquad \mathbf{F} = \nabla f \;\Leftrightarrow\; \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path-independent} \;\Leftrightarrow\; \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed } C$$

Unfortunately, these equivalent formulations don't give us any effective way of deciding if a given field $\mathbf{F}$ is a conservative field or not. However, if we assume that $\mathbf{F}$ is not just continuous but is even continuously differentiable (meaning: M_x, M_y, N_x, N_y all exist and are continuous), then there is a simple and elegant criterion for deciding whether or not $\mathbf{F}$ is a gradient field in some region.

Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a region D. Then, in D,

$$(2) \qquad\qquad \mathbf{F} = \nabla f \text{ for some } f(x, y) \quad \Rightarrow \quad M_y = N_x \ .$$

Proof. Since $\mathbf{F} = \nabla f$, this means

$$
\begin{aligned}
M &= f_x &&\text{and} &N &= f_y \ . &&\text{Therefore,} \\
M_y &= f_{xy} &&\text{and} &N_x &= f_{yx} \ .
\end{aligned}
$$

But since these two mixed partial derivatives are continuous (since M_y and N_x are, by hypothesis), a standard 18.02 theorem says they are equal. Thus $M_y = N_x$. $\qquad\square$

This theorem may be expressed in a slightly different form, if we define the scalar function called the **two-dimensional curl** of $\mathbf{F}$ by

$$(3) \qquad\qquad\qquad\qquad \operatorname{curl} \mathbf{F} \;=\; N_x - M_y \ .$$

Then (2) becomes

$$(2') \qquad\qquad\qquad\qquad \mathbf{F} = \nabla f \quad \Rightarrow \quad \operatorname{curl} \mathbf{F} = 0 \ .$$

This criterion allows us to test $\mathbf{F}$ to see if it is a gradient field. Naturally, we would also like to know that the converse is true: if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is a gradient field. As we shall see, however, this requires some additional hypotheses on the region D. For now, we will assume D is the whole plane. Then we have

Converse to Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable for all x, y.

$$(4) \qquad M_y = N_x \text{ for all } x, y \quad \Rightarrow \quad \mathbf{F} = \nabla f \text{ for some differentiable } f \text{ and all } x, y.$$

The proof of (4) will be postponed until we have more technique. For now we will illustrate the use of the criterion and its converse.

1

Example 1. For which value(s), if any of the constants a, b will $axy\,\mathbf{i} + (x^2 + by)\,\mathbf{j}$ be a gradient field?

Solution.   The partial derivatives are continuous for all x, y and $M_y = ax$, $N_x = 2x$. Thus by (2) and (4), $\mathbf{F} = \nabla f \iff a = 2$; b is arbitrary.

Example 2. Are the fields $\mathbf{F} \;=\; \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$, $\mathbf{G} \;=\; \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$ conservative?

Solution.   We have (the second line follows from the first by interchanging x and y):

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) \;=\; \frac{-x^2 + y^2}{(x^2 + y^2)^2};\qquad \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right) \;=\; \frac{2xy}{(x^2 + y^2)^2};$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) \;=\; \frac{x^2 - y^2}{(x^2 + y^2)^2};\qquad \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) \;=\; \frac{2yx}{(x^2 + y^2)^2};$$

from this, we see immediately that

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right);\qquad \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right);$$

the two equations in the last line show respectively that $\mathbf{F}$ and $\mathbf{G}$ satisfy the criterion (2). However, neither field is defined at $(0,0)$, so that the converse (4) is not applicable. So the question cannot be decided just on the basis of (2) and (4). In fact, it turns out that $\mathbf{F}$ is a gradient field, since one can check that

$$\mathbf{F} \;=\; \nabla \ln \sqrt{x^2 + y^2} \;=\; \nabla \ln r\;.$$

On the other hand, $\mathbf{G}$ is not conservative, since if C is the unit circle $x = \cos t$, $y = \sin t$, we have

$$\oint_C \mathbf{G} \cdot d\mathbf{r} \;=\; \oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} \;=\; \int_0^{2\pi} \frac{\sin^2 t\,dt + \cos^2 t\,dt}{\sin^2 t + \cos^2 t} \;=\; 2\pi \;\neq\; 0.$$

We will return to this example later on.

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