18.02 Problem Set 8, Part II Solutions

Problem 1

a) $\boxed{\mathbf{F} = (1 - x - y)\mathbf{j}.}$ (See diagram (a) below.)

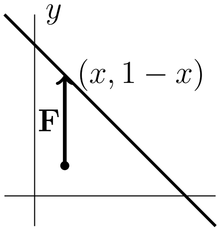


diagram a

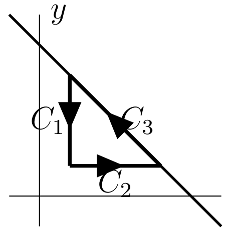


diagram b(i)

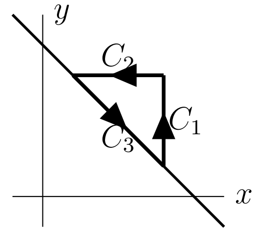


diagram b(ii)

b) The two possibilities for C are shown in diagrams b(i) and b(ii).

On C_3: (in both cases) $\mathbf{F} = 0 \ \Rightarrow \ \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$

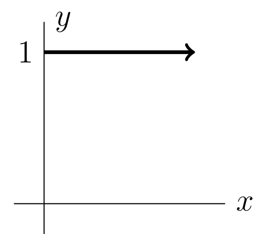
On C_2: (in both cases) $\mathbf{F} \cdot \mathbf{T} = 0 \ \Rightarrow \ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0.$

On C_1: case b(i) $\mathbf{F} \cdot \mathbf{T} < 0 \ \Rightarrow \ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds < 0.$

case b(i) $\mathbf{F} \cdot \mathbf{T} > 0 \ \Rightarrow \ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds > 0.$

Therefore in both cases $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 + C_2 + C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq 0.$

Problem 2

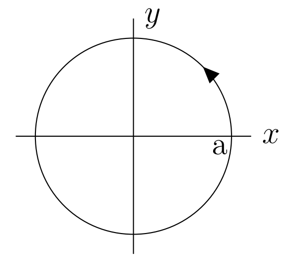


Path for problem 2a

a) Work $= \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\frac{x}{x^2+y^2}\, dx - \frac{y}{x^2+y^2}\, dy$

Path: $x = x$, $y = 1$, where $0 \le x < \infty$

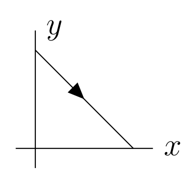
$$\Rightarrow \text{ work } = \int_0^\infty -\frac{x}{x^2+1}\, dx = -\frac{1}{2}\ln(x^2+1)\Big|_0^\infty = \boxed{-\infty.}$$



Path for problem 2b

b) Path: $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$

$$\Rightarrow \text{ work } = \int_0^{2\pi} -\frac{a\cos t}{a^2}(-a\sin t)\, dt - \frac{1\sin t}{a^2}(a\cos t)\, dt = \int_0^{2\pi} 0\, dt = \boxed{0.}$$



Path for problem 2c

c) Path: $x = t$, $y = 1 - t$, $0 \le t \le 1$

$$\Rightarrow \text{ work } = \int_0^1 -\frac{t}{2t^2-2t+1}\, dt - \frac{1-t}{2t^2-2t+1}(-dt)$$

$$= -\int_0^1 \frac{2t-1}{2t^2-2t+1}\, dt = -\frac{1}{2}\ln(2t^2-2t+1)\Big|_0^1 = \boxed{0.}$$

Problem 3

a) $r = \sqrt{x^2+y^2} \ \Rightarrow \ \dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$, likewise $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

$$\Rightarrow \ -\boldsymbol{\nabla} \ln r = -\langle \frac{\partial \ln r}{\partial x}, \frac{\partial \ln r}{\partial y} \rangle = -\langle \frac{1}{r} \cdot \frac{x}{r}, \frac{1}{r} \cdot \frac{y}{r} \rangle = -\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle. \quad \text{QED}$$

b) $$\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = -\ln r\big|_{P_1}^{P_2} = -(\ln r_2 - \ln r_1) = \boxed{-\ln \frac{r_2}{r_1}.} \quad \text{QED}$$

Problem 4

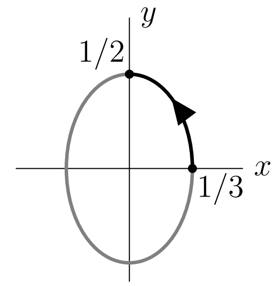
a) $\mathbf{F} = \langle 2xy + 2y^2, x^2 + 4xy \rangle \;\Rightarrow\; \int_c \mathbf{F} \cdot d\mathbf{r} = \boxed{\int_c (2xy + 2y^2)\, dx + (x^2 + 4xy)\, dy.}$

b) Path: $\quad x = \frac{1}{3}\cos t,\ y = \frac{1}{2}\sin t, \quad 0 \le t \le \pi/2.$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left(\frac{2}{6} \cos t \sin t + \frac{2}{4} \sin^2 t \right) \left(-\frac{1}{3} \sin t \right) dt$$

$$+ \left(\frac{1}{9} \cos^2 t + \frac{4}{6} \cos t \sin t \right) \left(\frac{1}{2} \cos t \right) dt$$

$$\boxed{= \int_0^{\pi/2} \left(-\frac{1}{9} \cos t \sin^2 t - \frac{1}{6} \sin^3 t + \frac{1}{18} \cos^3 t + \frac{1}{3} \cos^2 t \sin t \right) dt.}$$



Path for problem 4

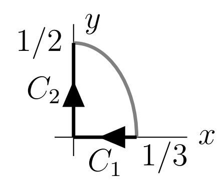
c) $\displaystyle\int_{(1/3,0)}^{(0,1/2)} \mathbf{F} \cdot d\mathbf{r} = x^2 y + 2xy^2 \Big|_{(1/3,0)}^{(0,1/2)} = \boxed{0.}$

d) By path independence: $\quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$

On C_1: $\quad y = 0,\ dy = 0 \;\Rightarrow\; (2xy + 2y^2)\, dx + (x^2 + 4xy)\, dy = 0.$

On C_2: $\quad x = 0,\ dx = 0 \;\Rightarrow\; (2xy + 2y^2)\, dx + (x^2 + 4xy)\, dy = 0.$

$\Rightarrow$ each of the integrals is 0 $\;\Rightarrow\; \boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0.}$



Path for problem 4d

Problem 5

a) $\operatorname{curl} \mathbf{F} = (N_x - M_y)\mathbf{k} = (3x^2 - x)\mathbf{k} \neq 0 \;\Rightarrow\; \mathbf{F}$ is not conservative.

b) We pretend $\mathbf{F}$ is conservative and look for a potential function f using method 1.
Since $\mathbf{F}$ is not conservative this will run into trouble.
Method 1. We use the path shown.

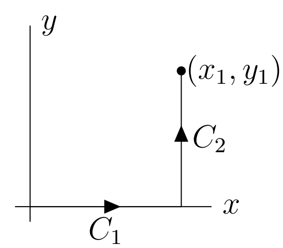
$$f(x_1, y_1) = \int_{C_1 + C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 + C_2} xy\, dx + x^3\, dy.$$

On C_1: $\quad y = 0,\ dy = 0 \;\Rightarrow\; xy\, dx + x^3\, dy = 0.$

On C_2: $\quad x = x_1,\ dx = 0 \;\Rightarrow\; xy\, dx + x^3\, dy = x_1^3\, dy.$

$$\Rightarrow\; f(x_1, y_1) = \int_0^{y_1} x_1^3\, dy = x_1^3 y_1 \;\Rightarrow\; f(x, y) = x^3 y.$$

So far so good, the trouble is $\boldsymbol{\nabla} f = \langle 3x^2 y, x^3 \rangle \neq \mathbf{F}$.



Path for problem 5b

c) Again pretending there is a potential function f.

Method 2: $f_x = xy \;\Rightarrow\; f = \dfrac{x^2 y}{2} + g(y).$

$f_y = \dfrac{x^2}{2} + g'(y) = x^3 \;\Rightarrow\; g'(y) = x^3 - \dfrac{x^2}{2}.$

The trouble here is that no possible function $g(y)$ can satisfy this equation.