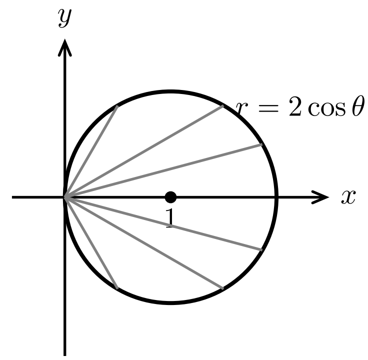
Double integration in polar coordinates

1. Compute $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ and R is the region inside the circle of radius 1, centered at (1,0).

<u>**Answer:**</u> First we sketch the region R



Both the integrand and the region support using polar coordinates. The equation of the circle in polar coordinates is $r = 2\cos\theta$, so using radial stripes the limits are

(inner) r from 0 to $2\cos\theta$; (outer) θ from $-\pi/2$ to $\pi/2$.

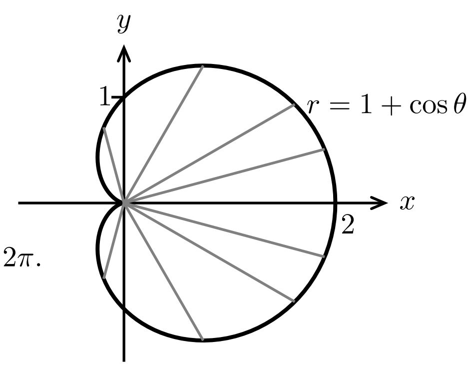
Thus,

$$\iint_R f(x,y)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} dr\,d\theta.$$

Inner integral: $2\cos\theta$.

Outer integral: $2\sin\theta\big|_{-\pi/2}^{\pi/2} = 4$.

2. Find the area inside the cardioid $r = 1+\cos\theta$.



<u>**Answer:**</u> The cardioid is so-named because it is heart-shaped.

Using radial stripes, the limits of integration are

(inner) r from 0 to $1+\cos\theta$; (outer) θ from 0 to 2π.

So, the area is

$$\iint_R dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta.$$

Inner integral: $\dfrac{(1+\cos\theta)^2}{2}$.

Side work:

$$\int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C \Rightarrow \int_0^{2\pi}\cos^2\theta\,d\theta = \pi.$$

Outer integral:

$$\int_0^{2\pi}\frac{(1+\cos\theta)^2}{2} = \int_0^{2\pi}\frac{1}{2} + \cos\theta + \frac{\cos^2\theta}{2}\,d\theta = \pi + 0 + \frac{\pi}{2} = \frac{3\pi}{2}.$$

The area of the cardioid is $\dfrac{3\pi}{2}$.

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