CHRISTINE BREINER: Welcome back to recitation. In this video, what I'd like us to do is answer some questions I've posed here in a really long problem. So let me take us through it.

So we're going to consider a position vector that's described by x of t, y of t, 0. I want to have it in three-space so I can take a cross product later. And then we're going to suppose that it actually has constant length. And that when I look at the acceleration vector at t, it's actually equal to a constant times r of t, where the constant is not 0. OK?

So I can, these are the two things I know about this position vector. It has constant length. For all t, it has constant length. And the acceleration is always equal to some constant times the position. OK, that's what I'm giving you.

And now I want you to use those things and vector differentiation to show that r dot v is equal to 0, where v is the velocity. And then to show that r cross v is constant. So you're going to have to figure out-- essentially, the thing you have to figure out is what relationship do you want to differentiate to show these two things. OK? That's the hard part of this problem.

And then, I went to see if you can give an example of such an r. So if you can give an example of a position vector that has these properties. And maybe if you're having a hard time, the first thing for you to do might be to think about this, and to see if you can figure out an example of that, and then see kind of how things work together in that example. That may actually proved helpful.

So why don't you work on this problem, pause the video, and then when you're feeling good about your answer, you can bring the video back up and I'll show you how I do it.

OK, welcome back. So there was a lot to do in this problem, but let me just remind you what the framework is. We have a position vector, and we know two things about it. We know that it has constant length and we know that the acceleration is always equal to some constant times the position. I didn't give you the constant, but we know it's always equal to some constant times the position.

And then we wanted to show two things using vector differentiation. We wanted to show that r dot v was 0. And we wanted to show that r cross v was constant. And then we want to talk about what is an example.

So let's start off and see if we can figure out how to show that r dot v is equal to 0. And, you know, as you're thinking about this problem, something that you want to remember as you're thinking about this is, well, what are the things that I know? I know that r dot r is constant, so I'm going to write that down. r dot r-- I'm not going to say is equal to c, because that's a different constant-- so I'll just, let me just call it c_1. OK? That's a different constant than my c, maybe.

OK, I know that r dot r is some constant, and I want to show something about r dot v. Right? So if I'm looking at this and I say, well, I know this thing here. So it's the only dot product relationship that I have. Because right now, I know a relationship between a and r, and I know r has constant length.

Since that's all I gave you, if I want to look at a dot product, this is the relationship I know. The constant length thing. And so I know I somehow have to use this one to figure out something about r dot v.

OK. Well, what's the point that we should realize? What is v? v is d/dt of r. So I could take the derivative-- if I could take the derivative of just one of these things, then I would get r dot v down here. Right? If I took d/dt of just one of these, I would get the r dot v. And d/dt of this is 0.

But I can't do that, right? Because if I take d/dt of this whole thing, I'm going to end up having to differentiate this r once and leave this alone, and I'm going to have to differentiate this r and leave this alone. But if you heard what I was just saying, maybe you see that that's actually still going to be OK.

So let's look at what happens. I mean, this is all really I have to work with, so I'm going to explore. Let's look at what happens when I take d/dt of r dot r.

And I'm going to start leaving off the hats here, because I'm going to leave them off somewhere, so we'll just leave them off now, and then I won't leave some off and put some on. So from here on out, I'm just going to write r, v, and a without that hats, but they're vectors.

OK. So d/dt of r dot r, well, I have to take d/dt of r, and then I dot that with r, and then I take r dotted with-- sorry-- d/dt of r. Right? Well, what do I get here? d/dt of r we said was v, so that's v dot r. And what do I get here? This is r dot-- there's d/dt of r again-- so I get r dot v.

Now the great thing about the dot product is that if I switch the order of these two, it's still the same thing. So I can just write this as-- well, I'll switch the order of this one. So they both look like r dot v plus r dot v, and that means I get 2 r dot v. Right?

So d/dt of r dot r is actually 2 of r dotted with v-- the position dotted with the velocity. Now why is this going to help me? Because what do I know about this quantity r dot r? I know it's constant, right? So what is d/dt of a constant? d/dt of a constant is 0. So I actually started off with knowing this was 0.

So if I go through the whole chain, I see 0 is equal to-- let me put the 0 down here again-- 0 is equal to 2 r dot v, and so r dot v I know is equal to 0. What does that mean geometrically? That means r and v are orthogonal.

And where is v going to sit? Well, if I come back over to how I described r, r is in the xy-plane, right? The z-component is 0.

So if I differentiate-- if I take d/dt of r-- I'm going to have the derivative of x as a function of t. And then whatever y is as a function of t, I take that derivative. And this is still 0.

So v is going to sit in the xy-plane, and based on what we know so far, we know it's actually orthogonal to r. So they make a 90-degree angle at all times t. OK, and how did we do that again?

I just want to remind you, we knew one dot product relationship, that was r dot r was a constant. So we differentiated that and tried to see what happened. And the main point at the end of it, was that when I had a v dot r, I could rewrite it as an r dot v, and so I just end up with two of something that I want to know about. So that's the main idea of the first part.

Now the second part was I asked you to figure out something over here. We wanted to know that r cross v is constant. OK? And t, it's always the same. All right?

So let's think about if I want to show that for every t something is constant, I could show-- actually, I've sort of seen it already-- I could show that its derivative is 0. OK. So if r cross v is constant-- or if its derivative is 0, I should say-- then r cross v was constant. Right? If its derivative is 0 for all t, then r cross v is constant. Right? So that's really what we want to exploit here. So let's look at the idea.

So we want to-- again, let me write it down-- show r cross v is constant. And the strategy we're going to use is to do this we're going to show that d/dt of r cross v is equal to 0. Right? If we can show that, then this means that for all t it's the same. It's not changing in t. Right? So for all t, r cross v is going to be the same, so r cross v is going to be constant.

So the difference between the two problems was in the first problem you didn't quite know maybe what expression you needed to differentiate to find what you were looking for. Here, we know what we need to differentiate, but we have to make sure we understand-- to show this is constant-- when we differentiate, we should get 0. OK? So that's sort of a slightly different type of problem. And I'm asking you maybe from the other side here.

So let's now-- let's just see what we get on the left-hand side. So what's d/dt of r cross v? Well, d/dt of r is v, right? So we get v cross v for the first term. So take d/dt of r, we get v. We leave this v alone, and then we add to that, r cross d/dt of v-- what's d/dt of v? That's a. Right?

Now, let's take a look at this. Well, v cross v, v is pointing in the same direction as itself. So when you try and take a cross product of that, you know that the length of your vector should be the area of the parallelogram formed by these two vectors. But v is pointing in the same direction as itself, so there's no area there. That's a geometric interpretation of why this thing should be 0.

Another reason is that remember that your v cross v is going to include a sine theta term, where theta is the angle between the two vectors, right? That's another formula you have. And so when you look at the angle between this vector and itself, it's 0. And sine 0 is 0. So this is, in fact, 0 in that part. So if this is 0, then we get what we want.

Well, I only gave you one other bit of information in the problem. And if you remember, it was that a is always equal to a constant times r.

So I can rewrite this right-hand side as r cross a constant times r. And because of properties of these cross products, I can pull out the constant. Or I can actually, I guess I don't need to pull it out to talk about it, but it's nicer if I pull it out. And look at what I have here.

I have the exact same situation as v cross v. I mean, this is still pointing in the same direction. Constant times r and r still point in the same direction as if I were to compare r and r. So I didn't have to pull out the constant, but then right here it's very easy to see that this is also 0. So I had 0 plus this, so I get 0.

So, I've shown through this process-- maybe I should have written equal signs here-- that d/dt of r cross v is actually equal to 0. And so you see that this cross product between r and v--

which we know are orthogonal sitting in the xy plane-- that it's always the same, it's always the same vector.

And now I asked you to give an example, and maybe you thought of an example first, and then thought of how it worked. And so the easiest example is if you let r of t equal cosine t, sine t. So the easiest example-- there are others, obviously-- is if you let r of t equal cosine t, sine t comma 0. Sorry, I was thinking about it in three-space, right?

And in fact, if I were to scale this, it would still work. I could put any constant in front. This carves out-- if I let t go between 0 and 2 pi, or even minus infinity to infinity, I'm just carving out vectors that are-- the position vector is always on the unit circle on this case. Right? If I put a constant in front, it's on another circle. So for whatever values of t I'm letting myself vary over, all the vectors are going to lie on some part of a circle. OK?

And so this is maybe the easy example. Maybe you want to calculate, just to give yourself some practice, what v of t is and what a of t actually is. What these two quantities actually are, and then look at what happens. What happens with r and v, and see why r dot v is equal to 0.

What you should see-- I'll try and give you a picture of it geometrically. What you should see is that, you know, if this is-- we'll see if I can effectively do a two-dimensional drawing in three-space-- if this is the r I'm looking at, that v has to be coming in this way, out this way. So there's r and there's v. And this angle-- because I'm trying to squash what was a circle-- I guess I'll look from above first. My picture, from above, is looking like something like this. There's r and there's v, and that's a right angle. Right?

So if I look from-- coming down on to the xy-plane, here's a position vector, here's its velocity, they're orthogonal. And as I move all the way around the circle, that position vector and the velocity are going to keep that relationship.

When I look at this-- I'm trying to insert in the z-axis here. When I look at this, if I look at what is the cross product of these two vectors, well it's always going to point in the z-direction. It's going to point straight in the z-direction from here. Because it's orthogonal to both of these, it's going to point straight in the z-direction.

I know r is constant length. I can then see v is constant length from the example I have here. And as I rotate, I'm going to get that this is constant length. OK? So this is where the picture of what we were actually describing very much more generally in the first part of this.

So the main point that I want us to see in this problem is that when we want to find information about relationships between r, v, and a-- this position and velocity and acceleration-- what we can do is differentiate these vector fields. We saw an example when you were looking at I think Kepler's second law you saw this in lecture. But I just want to show you that you can use this if you don't necessarily know explicitly things about a position vector. You can still find out things about its relationship if you're given some information. You don't actually have to have a formula to find out some information about these relationships.

So I think that's where I'll stop.