Velocity and acceleration

Now we will see one of the benefits of using the position vector. Let's assume we have a moving point with position vector

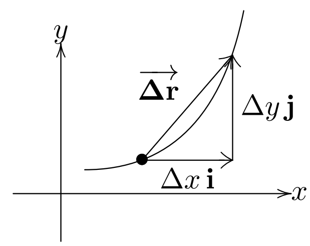
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

(We assume the point moves in the plane. The extension to a point moving in space is trivial.)

Velocity

Over a short time Δt the position changes by $\Delta\mathbf{r}$. The average velocity over this time is simply

$$\frac{\Delta\mathbf{r}}{\Delta t}, \text{ i.e., displacement/time.}$$



The figure shows $\Delta\mathbf{r} = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}$. Dividing by Δt we get

$$\text{average velocity } = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\mathbf{i} + \frac{\Delta y}{\Delta t}\,\mathbf{j}$$

Now, as we usually do in calculus, we let $\Delta t \to 0$. The average velocity becomes the (instantaneous) velocity and the ratios in the formula above become derivatives. For completeness we write the velocity vector in a number of different forms

$$\text{velocity } = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = x'\mathbf{i} + y'\mathbf{j} = \langle x', y'\rangle.$$

Tangent vector: (same thing as velocity)

In the picture above, we see that as Δt shrinks to 0 the vector $\frac{\Delta\mathbf{r}}{\Delta t}$ becomes tangent to the curve. When the parameter is time we can rightfully refer to $\mathbf{r}'(t)$ as the *velocity*. In general, we will abuse the language and refer to the derivative of position with respect to any parameter as velocity. If we are thinking geometrically or want to be precise, we will call the derivative by its geometric name: the *tangent vector*.

As always, we encourage you to remember the geometric view of the velocity vector. Knowing it is tangent to the curve will be important as we develop the subject and solve problems.

Acceleration

There is no reason to stop taking derivatives after one. Since acceleration is change in velocity per unit time, we get

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} = \langle x'', y''\rangle.$$

Example: A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = v_{0,x}\,t\,\mathbf{i} + (-\frac{g}{2}t^2 + v_{0,y}t)\mathbf{j}.$$

Find its velocity and acceleration vectors.

Answer:

$$\text{velocity } = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = v_{0,x}\mathbf{i} + (-gt + v_{0,y})\mathbf{j},$$

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -g\mathbf{j}.$$

Example: Find the velocity and acceleration vectors for the cycloid

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$$

Answer: As noted, this a slight abuse of language,

$$\text{velocity } = \text{tangent vector } = \mathbf{v}(\theta) = \frac{d\mathbf{r}}{d\theta} = \langle x'(\theta), y'(\theta)\rangle = \langle 1 - \cos\theta, \sin\theta\rangle.$$

$$\text{acceleration } = \mathbf{a}(\theta) = \frac{d\mathbf{v}}{d\theta} = \langle \sin\theta, \cos\theta\rangle.$$

Example: In the cycloid above, suppose the wheel rolls at 3 revolutions per second. Write the parametric equations in terms of time, and compute the velocity.

Answer: Since 3 rev/second = 6π radians/sec, we have $\theta = 6\pi t$. Therefore,

$$x(t) = 6\pi t - \sin(6\pi t), \quad y(t) = 1 - \cos(6\pi t).$$

$$\mathbf{v}(t) = \langle 6\pi - 6\pi\cos(6\pi t), 6\pi\sin(6\pi t)\rangle.$$