32. Stokes Theorem

Definition 32.1. *We say that a vector field*

$$\vec{F}\colon A \longrightarrow \mathbb{R}^m,$$

has ***compact support*** *if there is a compact subset $K \subset A$ such that*

$$\vec{F}(\vec{x}) = \vec{0},$$

for every $\vec{x} \in A - K$.

If $S \subset \mathbb{R}^3$ is a smooth manifold (possibly with boundary) then we will call S a *surface*. An *orientation* is a "continuous" choice of unit normal vector. Not every surface can be oriented. Consider for example the Möbius band, which is obtained by taking a piece of paper and attaching it to itself, except that we add a twist.

Theorem 32.2 (Stokes' Theorem)**.** *Let $S \subset \mathbb{R}^3$ be a smooth oriented surface with boundary and let $\vec{F}\colon S \longrightarrow \mathbb{R}^3$ be a smooth vector field with compact support.*

Then

$$\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \int_{\partial S} \vec{F} \cdot \mathrm{d}\vec{s},$$

where ∂S is oriented compatibly with the orientation on S.

Example 32.3. *Let S be a smooth 2-manifold that looks like a pair of pants. Choose the orientation of S such that the normal vector is pointing outwards. There are three oriented curves C_1, C_2 and C_3 (the two legs and the waist). Suppose that we are given a vector field $\vec{B}$ with zero curvature. Then* (32.2) *says that*

$$\int_{C_3} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_1'} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_2'} \vec{B} \cdot \mathrm{d}\vec{s} = \iint_S \operatorname{curl} \vec{B} \cdot \mathrm{d}\vec{S} = 0.$$

Here C_1' and C_2' denote the curves C_1 and C_2 with the opposite orientation. In other words,

$$\int_{C_3} \vec{B} \cdot \mathrm{d}\vec{s} = \int_{C_1} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_2} \vec{B} \cdot \mathrm{d}\vec{s}.$$

Proof of (32.2)*.* We prove this in three steps, in very much the same way as we proved Green's Theorem.

Step 1: We suppose that $M = \mathbb{H}^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, where the plane is the xy-plane. In this case, we can take $\hat{n} = \hat{k}$, and this induces the standard orientation of the boundary. Note that

$$\operatorname{curl} \vec{F} \cdot \hat{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

and so the result reduces to Green's Theorem. This completes step 1.

Step 2: We suppose that there is a compact subset $K \subset S$ and a parametrisation

$$\vec{g}\colon \mathbb{H}^2 \cap U \longrightarrow S \cap W,$$

which is compatible with the orientation, such that

(1) $\vec{F}(\vec{x}) = \vec{0}$ if $\vec{x} \in S - K$, and
(2) $K \subset S \cap W$.

Define a vector field $\vec{G}\colon \mathbb{H}^2 \longrightarrow \mathbb{R}^2$ by the rule

$$\vec{G}(u,v) = \begin{cases} \vec{F}(\vec{g}(u,v)) \cdot D\vec{g}(u,v) & (u,v) \in U \\ \vec{0} & (u,v) \notin U. \end{cases}$$

Note that

$$\begin{aligned} G_1(u,v) &= F_1\frac{\partial x}{\partial u} + F_2\frac{\partial y}{\partial u} + F_3\frac{\partial z}{\partial u} \\ G_2(u,v) &= F_1\frac{\partial x}{\partial v} + F_2\frac{\partial y}{\partial v} + F_3\frac{\partial z}{\partial v}. \end{aligned}$$

Using step 1, it is enough to prove:

Claim 32.4.

(1)
$$\iint_S \operatorname{curl}\vec{F}.\mathrm{d}\vec{S} = \iint_{\mathbb{H}^2} \left(\frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v.$$

(2)
$$\iint_{\partial S} \vec{F}\cdot \mathrm{d}\vec{s} = \iint_{\partial\mathbb{H}^2} \vec{G}\cdot \mathrm{d}s.$$

Proof of (32.4). Note that

$$\begin{aligned} \operatorname{curl}\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial \vec{g}}{\partial u} &= \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k} \\ \frac{\partial \vec{g}}{\partial v} &= \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}. \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \\
&= \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} - \frac{\partial(x,z)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}.
\end{aligned}$$

So,

$$\operatorname{curl} \vec{F} \cdot \frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \frac{\partial(y,z)}{\partial(u,v)} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \frac{\partial(x,z)}{\partial(u,v)} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \frac{\partial(x,y)}{\partial(u,v)}.$$

On the other hand, if one looks at the proof of the second step of Green's theorem, we see that

$$\frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v},$$

is also equal to the RHS (in fact, what we calculated in the proof of Green's theorem was the third term of the RHS; by symmetry the other two terms have the same form). This is (1).

For (2), let's parametrise $\partial \mathbb{H}^2 \cap U$ by $\vec{x}(u) = (u, 0)$ and $\partial S \cap W$ by $\vec{s}(u) = \vec{g}(\vec{x}(u))$. Then

$$\begin{aligned}
\int_{\partial S} \vec{F} \cdot \mathrm{d}\vec{s} &= \int_{\partial S \cap W} \vec{F} \cdot \mathrm{d}\vec{s} \\
&= \int_a^b \vec{F}(\vec{s}(u)) \cdot \vec{s}'(u)\,\mathrm{d}u \\
&= \int_a^b \vec{F}(\vec{g}(\vec{x}(u))) \cdot D\vec{g}(\vec{x}(u))\vec{x}'(u)\,\mathrm{d}u \\
&= \int_a^b \vec{G}(\vec{x}(u)) \cdot \vec{x}'(u)\,\mathrm{d}u \\
&= \int_{\partial \mathbb{H}^2 \cap U} \vec{G}. \cdot \mathrm{d}\vec{s} \\
&= \int_{\partial \mathbb{H}^2} \vec{G}. \cdot \mathrm{d}\vec{s},
\end{aligned}$$

and this is (2). □

This completes step 2.

Step 3: We again use partitions of unity. It is straightforward to cover the bounded set K by finitely many compact subsets $K_1, K_2, \dots, K_k$, such that given any smooth vector field which is zero outside K_i, then the conditions of step 2 hold. By using a partition of unity, we can find smooth functions $\rho_1, \rho_2, \ldots, \rho_k$ such that ρ_i is zero outside K_i and

$$1 = \sum_{i=1}^{k} \rho_i.$$

Multiplying both sides of this equation by $\vec{F}$, we have

$$\vec{F} = \sum_{i=1}^{k} \vec{F}_i,$$

where $\vec{F}_i = \rho_i \vec{F}$ is a smooth vector field, which is zero outside K_i. In this case

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} &= \sum_{i=1}^{k} \iint_S \operatorname{curl} \vec{F}_i \cdot \mathrm{d}\vec{S} \\
&= \sum_{i=1}^{k} \int_{\partial M} \vec{F}_i \cdot \mathrm{d}\vec{s} \\
&= \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s}. \qquad \square
\end{aligned}$$