PRACTICE FINAL
MATH 18.022, MIT, AUTUMN 10

You have three hours. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 10 problems, and the total number of points is 200. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20pts) Find the shortest distance between the line l given parametrically by

$$(x, y, z) = (1 + 2t, -3 + t, 2 - t),$$

and the intersection of the two planes Π_1 and Π_2 given by the equations

$$x + y + z = 3 \qquad \text{and} \qquad x - 2y + 3z = 2.$$

- Line where Π_1 and Π_2 intersect is

$$(x,y,z) = (1,1,1) + t\,(5,-2,-3) = (1+5t,\ 1-2t,\ 1-3t)$$

point satisfying both plane equations, found by inspection

vector perpendicular to $(1,1,1)$ and $(1,-2,3)$

$$(1,1,1) \times (1,-2,3) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (5,-2,-3)$$

- Shortest distance between two skew lines is

$$d = \left\| \operatorname{proj}_{\vec{n}} \overrightarrow{B_1B_2} \right\| = \left\| \left(\frac{\vec{n} \cdot \overrightarrow{B_1B_2}}{\vec{n} \cdot \vec{n}} \right) \vec{n} \right\| = \left\| \frac{13}{107} (-5,1,-9) \right\| = \boxed{\frac{13}{\sqrt{107}}}$$

vector perpendicular to $(2,1,-1)$ and $(5,-2,-3)$

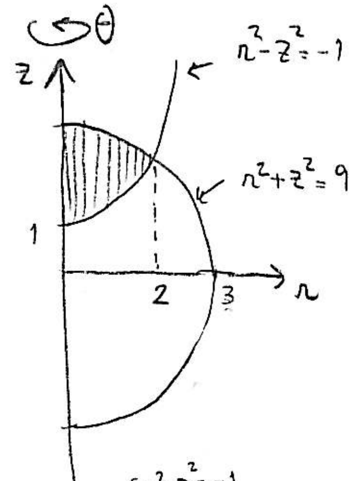
$$\overrightarrow{B_1B_2} = (1,1,1) - (1,-3,2) = (0,4,-1)$$

$$(2,1,-1) \times (5,-2,-3) = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 5 & -2 & -3 \end{vmatrix}$$

$$= (-5,1,-9)$$

2. (20pts) Let W be the solid inside the sphere $x^2+y^2+z^2=9$ and above the top sheet of the hyperboloid $x^2+y^2-z^2=-1$.

(a) Set up an integral in cylindrical coordinates for evaluating the volume of W.



$$\text{vol } W = \int_0^{2\pi}\int_0^{2}\int_{\sqrt{r^2+1}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$\begin{cases} r^2-z^2=-1 \\ r^2+z^2=9 \end{cases} \Rightarrow 2r^2=8 \Rightarrow r=2$$

(b) Evaluate this integral.

$$\text{vol } W = 2\pi\int_0^2 \left(\sqrt{9-r^2}-\sqrt{r^2+1}\right) r\,dr$$

$$= -\frac{2\pi}{3}\left((9-r^2)^{3/2}+(r^2+1)^{3/2}\right)\Big|_{r=0}^{2}$$

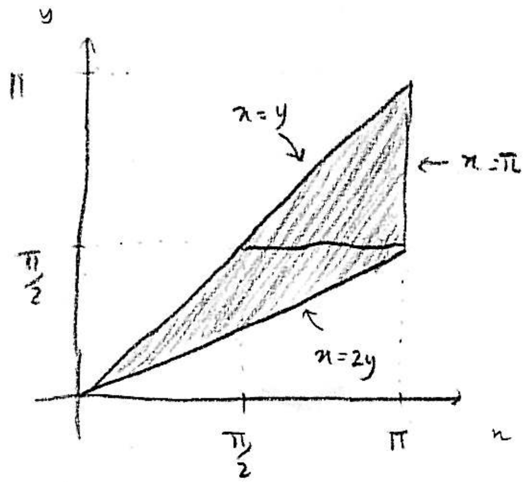
$$= -\frac{2\pi}{3}\left(5^{3/2}+5^{3/2}-9^{3/2}-1^{3/2}\right)$$

$$= -\frac{2\pi}{3}\left(2\sqrt{5}^3-28\right)$$

$$= \boxed{\frac{4\pi}{3}\left(14-\sqrt{5}^3\right)}$$

3. (20pts) Evaluate

$$\int_0^{\pi/2}\left(\int_y^{2y}\frac{\sin x}{x}\,\mathrm{d}x\right)\,\mathrm{d}y+\int_{\pi/2}^{\pi}\left(\int_y^{\pi}\frac{\sin x}{x}\,\mathrm{d}x\right)\,\mathrm{d}y.$$



cannot integrate directly

Change order of integration:

$$\int_0^{\pi}\int_{x/2}^{x}\frac{\sin x}{x}\,dy\,dx=\int_0^{\pi}\frac{\sin x}{x}\left(x-\frac{x}{2}\right)dx$$

$$=\int_0^{\pi}\frac{\sin x}{2}\,dx$$

$$=\frac{1}{2}\left(-\cos x\right|_{x=0}^{\pi}$$

$$=\frac{1}{2}(1+1)$$

$$=1$$

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{4}{9}\hat{\imath} - \frac{7}{9}\hat{\imath} - \frac{4}{9}\hat{k}, \quad \vec{N}(a) = \frac{1}{9}\hat{\imath} - \frac{4}{9}\hat{\imath} + \frac{8}{9}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = -\frac{4}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{1}{3}\hat{k}.$$

Find:

(i) the unit binormal vector $\vec{B}(a)$.

$$\vec{B}(a) = \vec{T}(a) \times \vec{N}(a) = \begin{vmatrix} i & j & k \\ 4/9 & -7/9 & -4/9 \\ 1/9 & -4/9 & 8/9 \end{vmatrix} = \frac{1}{81}(-56-16, -4-32, -16+7) = \boxed{\left(-\frac{8}{9}, -\frac{4}{9}, -\frac{1}{9}\right)}$$

(ii) the curvature $\kappa(a)$.

$$\vec{N}'(a) = -\kappa(a)\vec{T}(a) + \tau(a)\vec{B}(a), \text{ so } \vec{N}'(a)\cdot\vec{T}(a) = -\kappa(a)$$

$$\kappa(a) = -\left(-\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)\cdot\left(\frac{4}{9}, -\frac{7}{9}, -\frac{4}{9}\right) = \frac{16}{27} + \frac{7}{27} + \frac{4}{27} = \frac{27}{27} = \boxed{1}$$

(iii) the torsion $\tau(a)$.

$$\vec{N}'(a) = -\kappa(a)\vec{T}(a) + \tau(a)\vec{B}(a), \text{ so } \vec{N}'(a)\cdot\vec{B}(a) = \tau(a)$$

$$\tau(a) = \left(-\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)\cdot\left(-\frac{8}{9}, -\frac{4}{9}, -\frac{1}{9}\right) = \frac{32}{27} - \frac{4}{27} - \frac{1}{27} = \frac{27}{27} = \boxed{1}$$

5. (20pts) Let

$$S = \{ (x,y,z) \in \mathbb{R}^3 \mid xy^2z^3 - x^2y^2z^2 + x^3y^3 = 1 \}.$$

(a) Show that in a neighbourhood of the point $P = (1,1,1)$, the subset S is the graph of a smooth function $z = f(x,y)$.

$F(x,y,z) = xy^2z^3 - x^2y^2z^2 + x^3y^3$

$\frac{\partial F}{\partial z}(x,y,z) = 3xy^2z^2 - 2x^2y^2z$

$\left|\frac{\partial F}{\partial z}(1,1,1)\right| = 3-2 = 1 \neq 0$ so by the implicit function theorem, on a neighborhood of $(1,1,1)$, S is the graph of a function $z = f(x,y)$

(b) Find the derivative $Df(1,1)$.

On a neighborhood of $(x,y)=(1,1)$, we have $F(x,y,f(x,y)) = 1$.

Applying $\frac{\partial}{\partial x}$ on both sides,

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial x} = 0$, at $(1,1)$ is $2 + 1\,\frac{\partial f}{\partial x}(1,1) = 0 \Rightarrow \frac{\partial f}{\partial x}(1,1) = -2$

Applying $\frac{\partial}{\partial y}$ on both sides,

$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial y} = 0$, at $(1,1)$ is $3 + 1\,\frac{\partial f}{\partial y}(1,1) = 0 \Rightarrow \frac{\partial f}{\partial y}(1,1) = -3$

So $\boxed{Df(1,1) = (-2,-3)}$

Auxiliary computations.

$\frac{\partial F}{\partial x}(x,y,z) = y^2z^3 - 2xy^2z^2 + 3x^2y^3$

$\frac{\partial F}{\partial y}(x,y,z) = 2xyz^3 - 2x^2yz^2 + 3x^3y^2$

6. (20pts) Let K be the solid bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 3z = 12$ and let $f\colon K \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = xyz$.

(a) Show that f has a global maximum on K.

K is a compact set (it is closed and bounded), and f is a continuous function, so f must have a global maximum (and minimum) on K.

(b) Find this global maximum value of f on K.

$Df = (yz, xz, xy)$ does not vanish on the interior of K, so max must occur on the walls. On $x=0$, $y=0$, $z=0$ walls, $f(x,y,z)=0$, which is not the max (f attains positive values), so we use Lagrange multiplers to find maximum of f on $x+2y+3z=12$.

$$F(x,y,z) = xyz + \lambda(x+2y+3z-12)$$

$$\begin{cases} \dfrac{\partial F}{\partial x} = yz+\lambda = 0 \\ \dfrac{\partial F}{\partial y} = xz + 2\lambda = 0 \\ \dfrac{\partial F}{\partial z} = xy + 3\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x+2y+3z-12 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = -yz \\ (x-2y)z = 0 \\ (x-3z)y = 0 \\ x+2y+3z = 12 \end{cases} \Rightarrow \begin{cases} x = 2y \\ x = 3z \\ 3x = 12 \end{cases}$$

we don't want x, y or $z = 0$ because it would give $f = 0$, which is not a max

$$\Leftrightarrow x = 4,\ y = 2,\ z = \frac{4}{3}$$

$$\boxed{f\left(4,2,\frac{4}{3}\right) = \frac{32}{3}}$$

7. (20pts) Let D be the region bounded by the four curves $xy = 1$, $xy = 3$, $2y = x^2$ and $y = x^2$.
(a) Compute $\mathrm{d}x\,\mathrm{d}y$ in terms of $\mathrm{d}u\,\mathrm{d}v$, where $u = xy$ and $v = x^2/y$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix} = \left| -\frac{x^2}{y} - \frac{2x^2}{y} \right| = \frac{3x^2}{y} = 3v$$

$$\boxed{dx\,dy = \frac{1}{3v}\,du\,dv}$$

(b) Find the area of D.

$$\text{area } D = \int_D dx\,dy$$

$$= \int_1^2 \int_1^3 \frac{1}{3v}\,du\,dv$$

$$= \frac{1}{3}\,2\,\left(\log v\Big|_{v=1}^{2}\right)$$

$$= \boxed{\frac{2}{3}\log 2}$$

8. (20pts) Find the line integral of the vector field

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}, \quad = M\hat{i} + N\hat{j}$$

along the following oriented curves:
(i) The circle C_1 with equation $x^2+y^2=1$, oriented counter-clockwise.

$$\oint_{C_1} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt = \boxed{2\pi}$$

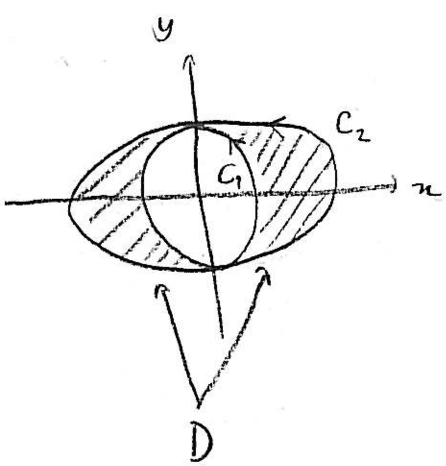
C_1 parametrized by $\vec{r}(t) = (\cos t, \sin t)$; $t \in [0, 2\pi]$

$\vec{r}\,'(t) = (-\sin t, \cos t)$

$\vec{F}(\vec{r}(t)) = \left(\frac{-\sin t}{1}, \frac{\cos t}{1}\right) = (-\sin t, \cos t)$

(ii) The ellipse C_2 with equation $x^2+2y^2=4$, oriented counter-clockwise.

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$



$$0 = \iint_D \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{=0} dA = \underbrace{\oint_{C_1} \vec{F}\cdot d\vec{s}}_{=2\pi} - \oint_{C_2} \vec{F}\cdot d\vec{s}$$

$$\Rightarrow \boxed{\oint_{C_2} \vec{F}\cdot d\vec{s} = 2\pi}$$

9. (20pts) Find the surface scalar integral

$$\iint_S (x^2 + y^2)\,\mathrm{d}S,$$

where S is the sphere of radius a centred at the origin.

$$\iint_S (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi} \underbrace{a^2 \sin^2\varphi}_{=r^2}\ \underbrace{a^2 \sin\varphi\, d\varphi\, d\theta}_{dS \text{ for a spherical surface}}$$

$$= 2\pi\, a^4 \int_0^{\pi} \sin^3\varphi\, d\varphi$$

$$= 2\pi\, a^4 \int_0^{\pi} (1-\cos^2\varphi)\sin\varphi\, d\varphi$$

$$= 2\pi\, a^4 \left(\left(-\cos\varphi\right|_{\varphi=0}^{\pi} + \left(\frac{\cos^3\varphi}{3}\right|_{\varphi=0}^{\pi} \right)$$

$$= 2\pi\, a^4 \left((0+1) + \tfrac{1}{3}(0-1)\right)$$

$$= \boxed{\frac{4}{3}\pi a^4}$$

10. (20pts) A smooth vector field is defined on the whole of $\mathbb{R}^3$, except two lines L and M, which intersect at the point P. Suppose that $\operatorname{curl} \vec{F} = 0$, and that

$$\int_{C_1} \vec{F} \cdot d\vec{s} = -2, \qquad \int_{C_2} \vec{F} \cdot d\vec{s} = 3 \qquad \text{and} \qquad \int_{C_3} \vec{F} \cdot d\vec{s} = 1.$$

Find the following integrals. Give your reasons.

(a)

$$\int_{C_4} \vec{F} \cdot d\vec{s}.$$

$$0 = \iint_{S_{top}} \overbrace{\nabla \times \vec{F}}^{=0} \cdot d\vec{S} = -\oint_{C_1} \vec{F} \cdot d\vec{s} - \oint_{C_2} \vec{F} \cdot d\vec{s} + \oint_{C_4} \vec{F} \cdot d\vec{s} = 2 - 3 + \oint_{C_4} \vec{F} \cdot d\vec{s}$$

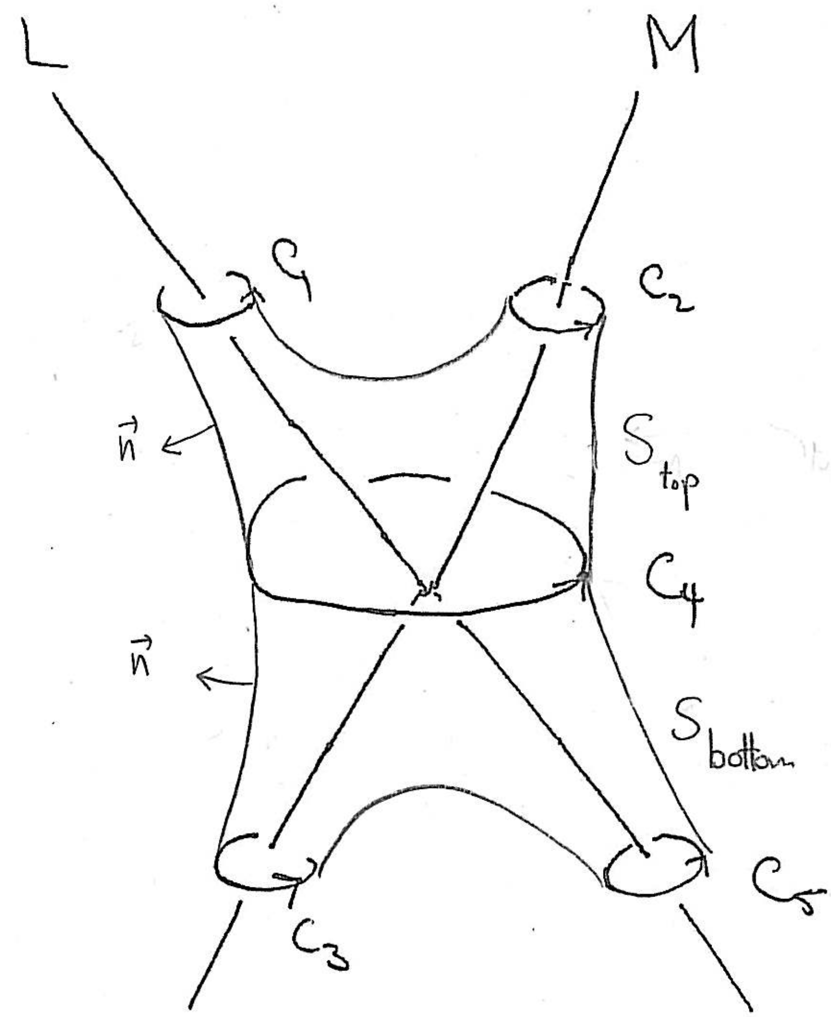
$$\Rightarrow \boxed{\oint_{C_4} \vec{F} \cdot d\vec{s} = 1}$$

(b)

$$\int_{C_5} \vec{F} \cdot d\vec{s}.$$

$$0 = \iint_{S_{bottom}} \overbrace{\nabla \times \vec{F}}^{=0} \cdot d\vec{S} = -\oint_{C_4} \vec{F} \cdot d\vec{s} + \oint_{C_3} \vec{F} \cdot d\vec{s} + \oint_{C_5} \vec{F} \cdot d\vec{s} = -1 + 1 + \oint_{C_5} \vec{F} \cdot d\vec{s}$$

$$\Rightarrow \boxed{\oint_{C_5} \vec{F} \cdot d\vec{s} = 0}$$


L
M
C_1
C_2
$\vec{n}$
S_{top}
C_4
$\vec{n}$
S_{bottom}
C_3
C_5