SECOND PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name:________________________________

Signature:________________________________

Recitation Time:________________________________

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Find a recursive formula for a sequence of points (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n), whose limit (x_∞, y_∞), if it exists, is a point of intersection of the curves

$$x^2 - y^2 = 1$$
$$x^2(x + 1) = y^2.$$

2. (20pts) Suppose that $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is differentiable at $P = (3, -2, 1)$ with derivative

$$DF(3, -2, 1) = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix}.$$

Suppose that $F(3, -2, 1) = (1, -3)$. Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = \|F(x, y, z)\|$.

(i) Show that the function $f(x, y, z)$ is differentiable at P.

(ii) Find $Df(3, -2, 1)$.

(iii) Find the directional derivative of f at P in the direction of $\hat{u} = -\frac{1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$.

3. (20pts) Let $F\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be a $\mathcal{C}^1$ function. Suppose that

$$DF(3,1,0,-1) = \begin{pmatrix} 1 & 3 & 1 & 3 \\ -1 & 2 & -1 & -2 \end{pmatrix}.$$

(a) Show that there is an open subset $U \subset \mathbb{R}^2$ containing $(3,1)$ and an open subset $V \subset \mathbb{R}^2$ containing $(0,-1)$ such that for all $(x,y) \in U$, the system of equations

$$F(x,y,z,w) = F(3,1,0,-1),$$

has the unique solution

$$(z,w) = (f_1(x,y), f_2(x,y)) \qquad \text{with} \qquad (z,w) \in V.$$

(b) Find the derivative $Df(3,1)$.

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{4}{9}\hat{\imath} - \frac{7}{9}\hat{\imath} - \frac{4}{9}\hat{k}, \quad \vec{B}(a) = \frac{1}{9}\hat{\imath} - \frac{4}{9}\hat{\imath} + \frac{8}{9}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = \hat{\imath} - 2\hat{\jmath}.$$

Find:

(i) the unit normal vector $\vec{N}(a)$.

(ii) the curvature $\kappa(a)$.

(iii) the torsion $\tau(a)$.

5. (20pts) Let $\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by $\vec{F}(x, y) = y\hat{\imath} + x\hat{\jmath}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

(ii) Is $\vec{F}$ incompressible?

(iii) Find a flow line that passes through the point $(1, 0)$.

(iv) Find a flow line that passes through the point (a, b), where $a^2 > b^2$.