

1. Vectors and Matrices

1A. Vectors

Definition. A **direction** is just a unit vector. The **direction of \mathbf{A}** is defined by

$$\text{dir } \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \quad (\mathbf{A} \neq \mathbf{0});$$

it is the unit vector lying along \mathbf{A} and pointed like \mathbf{A} (not like $-\mathbf{A}$).

1A-1 Find the magnitude and direction (see the definition above) of the vectors

a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ b) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ c) $3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

1A-2 For what value(s) of c will $\frac{1}{5}\mathbf{i} - \frac{1}{5}\mathbf{j} + c\mathbf{k}$ be a unit vector?

1A-3 a) If $P = (1, 3, -1)$ and $Q = (0, 1, 1)$, find $\mathbf{A} = PQ$, $|\mathbf{A}|$, and $\text{dir } \mathbf{A}$.

b) A vector \mathbf{A} has magnitude 6 and direction $(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})/3$. If its tail is at $(-2, 0, 1)$, where is its head?

1A-4 a) Let P and Q be two points in space, and X the midpoint of the line segment PQ . Let O be an arbitrary fixed point; show that as vectors, $OX = \frac{1}{2}(OP + OQ)$.

b) With the notation of part (a), assume that X divides the line segment PQ in the ratio $r : s$, where $r + s = 1$. Derive an expression for OX in terms of OP and OQ .

1A-5 What are the \mathbf{i} -components of a plane vector \mathbf{A} of length 3, if it makes an angle of 30° with \mathbf{i} and 60° with \mathbf{j} . Is the second condition redundant?

1A-6 A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the \mathbf{i} \mathbf{j} -components).

1A-7 Let $\mathbf{A} = a\mathbf{i} + b\mathbf{j}$ be a plane vector; find in terms of a and b the vectors \mathbf{A}' and \mathbf{A}'' resulting from rotating \mathbf{A} by 90° a) clockwise b) counterclockwise.

(Hint: make \mathbf{A} the diagonal of a rectangle with sides on the x and y -axes, and rotate the whole rectangle.)

c) Let $\mathbf{i}' = (3\mathbf{i} + 4\mathbf{j})/5$. Show that \mathbf{i}' is a unit vector, and use the first part of the exercise to find a vector \mathbf{j}' such that \mathbf{i}', \mathbf{j}' forms a right-handed coordinate system.

1A-8 The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a space vector, represented as an origin vector, and let α, β , and γ be the three angles ($\leq \pi$) that \mathbf{A} makes respectively with \mathbf{i} , \mathbf{j} , and \mathbf{k} .

a) Show that $\text{dir } \mathbf{A} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$. (The three coefficients are called the *direction cosines* of \mathbf{A} .)

b) Express the direction cosines of \mathbf{A} in terms of a, b, c ; find the direction cosines of the vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

c) Prove that three numbers t, u, v are the direction cosines of a vector in space if and only if they satisfy $t^2 + u^2 + v^2 = 1$.

1A-9 Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides **A** and **B**.)

1A-10 Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

1A-11 Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show $X = Y$.)

1A-12* Label the four vertices of a parallelogram in counterclockwise order as $OPQR$. Prove that the line segment from O to the midpoint of PQ intersects the diagonal PR in a point X that is $1/3$ of the way from P to R .

(Let $\mathbf{A} = OP$, and $\mathbf{B} = OR$; express everything in terms of \mathbf{A} and \mathbf{B} .)

1A-13* a) Take a triangle PQR in the plane; prove that as vectors $PQ + QR + RP = \mathbf{0}$.

b) Continuing part a), let \mathbf{A} be a vector the same length as PQ , but perpendicular to it, and pointing outside the triangle. Using similar vectors \mathbf{B} and \mathbf{C} for the other two sides, prove that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. (This only takes one sentence, and no computation.)

1A-14* Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn't cross itself (i.e., one whose interior is a single region); label its vertices P_1, P_2, \dots, P_n as you walk around it.

1A-15* Let P_1, \dots, P_n be the vertices of a regular n -gon in the plane, and O its center; show without computation or coordinates that $OP_1 + OP_2 + \dots + OP_n = \mathbf{0}$,

a) if n is even; b) if n is odd.

1B. Dot Product

1B-1 Find the angle between the vectors

a) $\mathbf{i} - \mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ b) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

1B-2 Tell for what values of c the vectors $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ will

a) be orthogonal b) form an acute angle

1B-3 Using vectors, find the angle between a longest diagonal PQ of a cube, and

a) a diagonal PR of one of its faces; b) an edge PS of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

1B-4 Three points in space are $P : (a, 1, -1)$, $Q : (0, 1, 1)$, $R : (a, -1, 3)$. For what value(s) of a will PQR be

a) a right angle b) an acute angle ?

1B-5 Find the component of the force $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in

a) the direction $\frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}$ b) the direction of the vector $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

1B-6 Let O be the origin, c a given number, and \mathbf{u} a given direction (i.e., a unit vector). Describe geometrically the locus of all points P in space that satisfy the vector equation

$$OP \cdot \mathbf{u} = c|OP|.$$

In particular, tell for what value(s) of c the locus will be

- a) a plane b) a ray (i.e., a half-line) c) empty

(Hint: divide through by $|OP|$.)

1B-7 a) Verify that $\mathbf{i}' = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$ and $\mathbf{j}' = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}}$ are perpendicular unit vectors that form a right-handed coordinate system

b) Express the vector $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j}$ in the $\mathbf{i}'\mathbf{j}'$ -system by using the dot product.

c) Do b) a different way, by solving for \mathbf{i} and \mathbf{j} in terms of \mathbf{i}' and \mathbf{j}' and then substituting into the expression for \mathbf{A} .

1B-8 The vectors $\mathbf{i}' = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$, $\mathbf{j}' = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$, and $\mathbf{k}' = \frac{\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{6}}$ are three mutually perpendicular unit vectors that form a right-handed coordinate system.

- a) Verify this. b) Express $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in this system (cf. 1B-7b)

1B-9 Let \mathbf{A} and \mathbf{B} be two plane vectors, neither one of which is a multiple of the other. Express \mathbf{B} as the sum of two vectors, one a multiple of \mathbf{A} , and the other perpendicular to \mathbf{A} ; give the answer in terms of \mathbf{A} and \mathbf{B} .

(Hint: let $\mathbf{u} = \text{dir } \mathbf{A}$; what's the \mathbf{u} -component of \mathbf{B} ?)

1B-10 Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

1B-11 Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal lengths.

1B-12 Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

1B-13 Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

(Hint: consider two unit vectors making angles θ_1 and θ_2 with the positive x -axis.)

1B-14 Prove the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos\theta$ by using the algebraic laws for the dot product and its geometric interpretation.

1B-15* The **Cauchy-Schwarz inequality**

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or in space:

$$(*) \quad |\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|.$$

Under what circumstances does equality hold?

b) If the vectors are plane vectors, write out what this inequality says in terms of $\mathbf{i}\mathbf{j}$ -components.

c) Give a different argument for the inequality (*) as follows (this argument generalizes to n -dimensional space):

- i) for all values of t , we have $(\mathbf{A} + t\mathbf{B}) \cdot (\mathbf{A} + t\mathbf{B}) \geq 0$;
- ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in t ;
- iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

1C. Determinants

1C-1 Calculate the value of the determinants a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

1C-2 Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

- a) the first row b) the first column

1C-3 Find the area of the plane triangle whose vertices lie at

- a) $(0, 0), (1, 2), (1, -1)$; b) $(1, 2), (1, -1), (2, 3)$.

1C-4 Show that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$.

(This type of determinant is called a **Vandermonde** determinant.)

1C-5 a) Show that the value of a 2×2 determinant is unchanged if you add to the second row a scalar multiple of the first row.

- b) Same question, with “row” replaced by “column”.

1C-6 Use a Laplace expansion and Exercise 5a to show the value of a 3×3 determinant is unchanged if you add to the second row a scalar multiple of the third row.

1C-7 Let (x_1, y_1) and (x_2, y_2) be two unit vectors. Find the maximum value of the function

$$f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

1C-8* The base of a parallelepiped is a parallelogram whose edges are the vectors \mathbf{b} and \mathbf{c} , while its third edge is the vector \mathbf{a} . (All three vectors have their tail at the same vertex; one calls them “coterminal”.)

- a) Show that the volume of the parallelepiped \mathbf{abc} is $\pm \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

b) Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$ the determinant whose rows are respectively the components of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(These two parts prove (3), the volume interpretation of a 3×3 determinant.

1C-9 Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices $(0, 0, 0)$, $(0, -1, 2)$, $(0, 1, -1)$, $(1, 2, 1)$. (The volume of a tetrahedron is $\frac{1}{3}(\text{base})(\text{height})$.)

1C-10 Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

1D. Cross Product

1D-1 Find $\mathbf{A} \times \mathbf{B}$ if

a) $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

1D-2 Find the area of the triangle in space having its vertices at the points

$$P : (2, 0, 1), \quad Q : (3, 1, 0), \quad R : (-1, 1, -1).$$

1D-3 Two vectors \mathbf{i}' and \mathbf{j}' of a right-handed coordinate system are to have the directions respectively of the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find all three vectors \mathbf{i}' , \mathbf{j}' , \mathbf{k}' .

1D-4 Verify that the cross product \times does not in general satisfy the associative law, by showing that for the particular vectors \mathbf{i} , \mathbf{i} , \mathbf{j} , we have $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{j})$.

1D-5 What can you conclude about \mathbf{A} and \mathbf{B}

a) if $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$; b) if $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$.

1D-6 Take three faces of a unit cube having a common vertex P ; each face has a diagonal ending at P ; what is the volume of the parallelepiped having these three diagonals as coterminous edges?

1D-7 Find the volume of the tetrahedron having vertices at the four points

$$P : (1, 0, 1), \quad Q : (-1, 1, 2), \quad R : (0, 0, 2), \quad S : (3, 1, -1).$$

Hint: volume of tetrahedron = $\frac{1}{6}$ (volume of parallelepiped with same 3 coterminous edges)

1D-8 Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

1D-9 Show that the area of a triangle in the xy -plane having vertices at (x_i, y_i) , for

$i = 1, 2, 3$, is given by the determinant $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Do this two ways:

a) by relating the area of the triangle to the volume of a certain parallelepiped

b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the 2×2 determinant that would normally be used to calculate the area.

1E. Equations of Lines and Planes

1E-1 Find the equations of the following planes:

- a) through $(2, 0, -1)$ and perpendicular to $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
- b) through the origin, $(1, 1, 0)$, and $(2, -1, 3)$
- c) through $(1, 0, 1)$, $(2, -1, 2)$, $(-1, 3, 2)$
- d) through the points on the x , y and z -axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
- e) through $(1, 0, 1)$ and $(0, 1, 1)$ and parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

1E-2 Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.

1E-3 Find in parametric form the equations for

- a) the line through $(1, 0, -1)$ and parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
- b) the line through $(2, -1, -1)$ and perpendicular to the plane $x - y + 2z = 3$
- c) all lines passing through $(1, 1, 1)$ and lying in the plane $x + 2y - z = 2$

1E-4 Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

1E-5 The line passing through $(1, 1, -1)$ and perpendicular to the plane $x + 2y - z = 3$ intersects the plane $2x - y + z = 1$ at what point?

1E-6 Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

(Hint: Let \mathbf{n} be the unit normal to the plane. and P be a point on the plane; consider the component of OP in the direction \mathbf{n} .)

1E-7* Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

(Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren't, it could be shortened).)

1F. Matrix Algebra

1F-1* Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$. Compute

- a) $B + C$, $B - C$, $2B - 3C$.
- b) AB , AC , BA , CA , BC^T , CB^T
- c) $A(B + C)$, $AB + AC$; $(B + C)A$, $BA + CA$

1F-2* Let A be an arbitrary $m \times n$ matrix, and let I_k be the identity matrix of size k . Verify that $I_m A = A$ and $A I_n = A$.

1F-3 Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1F-4* Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix $AB - BA$:

$$\text{a) } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$$

1F-5 a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3, A^n . b) Do the same for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1F-6* Let A, A', B, B' be 2×2 matrices, and O the 2×2 zero matrix. Express in terms of these five matrices the product of the 4×4 matrices $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$.

1F-7* Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Show there are no values of a and b such that $AB - BA = I_2$.

1F-8 a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A ?

b)* If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is the matrix A ?

1F-9 A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that

- a) each row of A is a row vector of length 1,
- b) two different rows are orthogonal vectors.

1F-10* Suppose A is a 2×2 orthogonal matrix, whose first entry is $a_{11} = \cos \theta$. Fill in the rest of A . (There are four possibilities. Use Exercise 9.)

1F-11* Show that if $A + B$ and AB are defined, then
 a) $(A + B)^T = A^T + B^T$, b) $(AB)^T = B^T A^T$.

1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation $A\mathbf{x} = \mathbf{b}$ by finding A^{-1} .

$$\text{1G-1* } A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}.$$

$$\text{1G-2* a) } A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \quad \text{b) } A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\text{1G-3 } A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}. \quad \text{Solve } A\mathbf{x} = \mathbf{b} \text{ by finding } A^{-1}.$$

1G-4 Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, \quad x_2 + x_3 = y_2 \quad -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i .

1G-5 Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.

1G-6* Another calculation of the inverse matrix.

If we know A^{-1} , we can solve the system $Ax = y$ for x by writing $x = A^{-1}y$. But conversely, if we can solve by some other method (elimination, say) for x in terms of y , getting $x = By$, then the matrix $B = A^{-1}$, and we will have found A^{-1} .

This is a good method if A is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

$$\text{a) Let } A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \text{find } A^{-1} \text{ by solving} \quad \begin{array}{l} -x_1 + x_2 + 3x_3 = y_1 \\ 2x_2 - x_3 = y_2 \\ x_3 = y_3 \end{array} \quad \text{for the } x_i$$

in terms of the y_i (start from the bottom and proceed upwards).

b) Calculate A^{-1} by the method given in the notes.

1G-7* Consider the rotation matrix $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ corresponding to rotation of the x and y axes through the angle θ . Calculate A_θ^{-1} by the adjoint matrix method, and explain why your answer looks the way it does.

1G-8* a) Show: A is an orthogonal matrix (cf. Exercise 1F-9) if and only if $A^{-1} = A^T$.

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if A and B are $n \times n$ orthogonal matrices, so is AB .

1G-9* a) Let A be a 3×3 matrix such that $|A| \neq 0$. The notes construct a right-inverse A^{-1} , that is, a matrix such that $A \cdot A^{-1} = I$. Show that every such matrix A also has a left inverse B (i.e., a matrix such that $BA = I$.)

(Hint: Consider the equation $A^T(A^T)^{-1} = I$; cf. Exercise 1F-11.)

b) Deduce that $B = A^{-1}$ by a one-line argument.

(This shows that the right inverse A^{-1} is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be I also.)

1G-10* Let A and B be two $n \times n$ matrices. Suppose that $B = P^{-1}AP$ for some invertible $n \times n$ matrix P . Show that $B^n = P^{-1}A^nP$. If $B = I_n$, what is A ?

1G-11* Repeat Exercise 6a and 6b above, doing it this time for the general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assuming $|A| \neq 0$.

1H. Cramer's Rule; Theorems about Square Systems

1H-1 Use Cramer's rule to solve for x in the following:

$$\begin{array}{ll} 3x - y + z = 1 & x - y + z = 0 \\ \text{(a)} \quad -x + 2y + z = 2, & \text{(b)} \quad x - z = 1. \\ x - y + z = -3 & -x + y + z = 2 \end{array}$$

1H-2 Using Cramer's rule, give another proof that if A is an $n \times n$ matrix whose determinant is non-zero, then the equations $Ax = 0$ have only the trivial solution.

$$x_1 - x_2 + x_3 = 0$$

1H-3 a) For what c -value(s) will $2x_1 + x_2 + x_3 = 0$ have a non-trivial solution?

$$-x_1 + cx_2 + 2x_3 = 0$$

b) For what c -value(s) will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$ have a non-trivial solution? (Write it as a system of homogeneous equations.)

c) For each value of c in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)* For each value of c in part (b), find a non-trivial solution to the corresponding system.

1H-4* Find all solutions to the homogeneous system
$$\begin{array}{l} x - 2y + z = 0 \\ x + y - z = 0 \\ 3x - 3x + z = 0 \end{array};$$
 use the method suggested in Exercise 3c above.

1H-5 Suppose that for the system $\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}$ we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Assume that $a_1 \neq 0$. Show that the system is consistent (i.e., has solutions) if and only if $c_2 = \frac{a_2}{a_1}c_1$.

1H-6* Suppose $|A| = 0$, and that \mathbf{x}_1 is a particular solution of the system $A\mathbf{x} = B$. Show that any other solution \mathbf{x}_2 of this system can be written as $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_0$, where \mathbf{x}_0 is a solution of the system $A\mathbf{x} = \mathbf{0}$.

1H-7 Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants a and b so that the function

$$f(x) = a \cos x + b \sin x$$

has prescribed values at two given x -values: $f(x_1) = y_1$, $f(x_2) = y_2$.

a) Show this is possible in one and only one way, if we assume that $x_2 \neq x_1 + n\pi$, for every integer n .

b) If $x_2 = x_1 + n\pi$ for some integer n , when can a and b be found?

1H-8* The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

$$\frac{ax + b}{(x - r_1)(x - r_2)} = \frac{c}{x - r_1} + \frac{d}{x - r_2}, \quad (a, b \text{ given}; c, d \text{ to be found});$$

what are the linear equations which determine the constants c and d ? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots r_i , $i = 1, 2, 3$. Evaluate the determinant by using column operations to get zeros in the top row.)

1I. Vector Functions and Parametric Equations

1I-1 The point P moves with constant speed v in the direction of the constant vector $a\mathbf{i} + b\mathbf{j}$. If at time $t = 0$ it is at (x_0, y_0) , what is its position vector function $\mathbf{r}(t)$?

1I-2 A point moves *clockwise* with constant angular velocity ω on the circle of radius a centered at the origin. What is its position vector function $\mathbf{r}(t)$, if at time $t = 0$ it is at

- (a) $(a, 0)$ (b) $(0, a)$

1I-3 Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞ . In each case, give the xy -equation of the curve along which P travels, and tell what part of the curve is actually traced out by P .

- a) $\mathbf{r} = 2\cos^2 t\mathbf{i} + \sin^2 t\mathbf{j}$ b) $\mathbf{r} = \cos 2t\mathbf{i} + \cos t\mathbf{j}$ c) $\mathbf{r} = (t^2 + 1)\mathbf{i} + t^3\mathbf{j}$
 d) $\mathbf{r} = \tan t\mathbf{i} + \sec t\mathbf{j}$

1I-4 A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O , and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P .

(Use vectors; express the position vector OP as a vector function of one variable.)

1I-5 A string is wound clockwise around the circle of radius a centered at the origin O ; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P .

(Use vectors; express the position vector OP as a vector function of one variable.)

1I-6 A bow-and-arrow hunter walks toward the origin along the positive x -axis, with unit speed; at time 0 he is at $x = 10$. His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity $\sqrt{5}$ in the first quadrant along the line $y = 2x$; at time 0 it is at the origin.

- a) Write down the vector function $\mathbf{A}(t)$ for the arrow at time t .
 b) The hunter shoots (and misses) when closest to the rabbit; when is that?

1I-7 The cycloid is the curve traced out by a fixed point P on a circle of radius a which rolls along the x -axis in the positive direction, starting when P is at the origin O . Find the vector function OP ; use as variable the angle θ through which the circle has rolled.

(Hint: begin by expressing OP as the sum of three simpler vector functions.)

1J. Differentiation of Vector Functions

1J-1 1. For each of the following vector functions of time, calculate the velocity, speed $|ds/dt|$, unit tangent vector (in the direction of velocity), and acceleration.

a) $e^t \mathbf{i} + e^{-t} \mathbf{j}$ b) $t^2 \mathbf{i} + t^3 \mathbf{j}$ c) $(1 - 2t^2) \mathbf{i} + t^2 \mathbf{j} + (-2 + 2t^2) \mathbf{k}$

1J-2 Let $OP = \frac{1}{1+t^2} \mathbf{i} + \frac{t}{1+t^2} \mathbf{j}$ be the position vector for a motion.

a) Calculate \mathbf{v} , $|ds/dt|$, and \mathbf{T} .

b) At what point in the speed greatest? smallest?

c) Find the xy -equation of the curve along which the point P is moving, and describe it geometrically.

1J-3 Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt} \mathbf{r} \cdot \mathbf{s} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt},$$

by calculating with components, letting $\mathbf{r} = x_1 \mathbf{i} + y_1 \mathbf{j}$ and $\mathbf{s} = x_2 \mathbf{i} + y_2 \mathbf{j}$.

1J-4 Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}.$$

Show that the velocity vector \mathbf{v} is always perpendicular to \mathbf{r} two different ways:

a) using the x, y, z -coordinates

b) without coordinates (use the formula in **1J-3**, which is valid also in space).

c) Prove the converse: if \mathbf{r} and \mathbf{v} are perpendicular, then the motion of P is on the surface of a sphere.

1J-5 a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**.)

b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point P moves with constant speed.

1J-6 For the helical motion $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}$,

a) calculate \mathbf{v} , \mathbf{a} , \mathbf{T} , $|ds/dt|$

b) show that \mathbf{v} and \mathbf{a} are perpendicular; explain using **1J-5**

1J-7 a) Suppose you have a differentiable vector function $\mathbf{r}(t)$. How can you tell if the parameter t is the arclength s (measured from some point in the direction of increasing t) without actually having to calculate s explicitly?

b) How should a be chosen so that t is the arclength if $\mathbf{r}(t) = (x_0 + at) \mathbf{i} + (y_0 + at) \mathbf{j}$?

c) How should a and b be chosen so that t is the arclength in the helical motion described in Exercise **1J-6**?

1J-8 a) Prove the formula $\frac{d}{dt} u(t)\mathbf{r}(t) = \frac{du}{dt} \mathbf{r}(t) + u(t)\frac{d\mathbf{r}}{dt}$.

(You may assume the vectors are in the plane; calculate with the components.)

b) Let $\mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j}$, the exponential spiral. Use part (a) to find the speed of this motion.

1J-9 A point P is moving in space, with position vector

$$\mathbf{r} = OP = 3 \cos t \mathbf{i} + 5 \sin t \mathbf{j} + 4 \cos t \mathbf{k}$$

- Show it moves on the surface of a sphere.
- Show its speed is constant.
- Show the acceleration is directed toward the origin.
- Show it moves in a plane through the origin.
- Describe the path of the point.

1J-10 The **positive curvature** κ of the vector function $\mathbf{r}(t)$ is defined by $\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$.

- Show that the helix of **1J-6** has constant curvature. (It is not necessary to calculate s explicitly; calculate $d\mathbf{T}/dt$ instead and relate it to κ by using the chain rule.)
- What is this curvature if the helix is reduced to a circle in the xy -plane?

1K. Kepler's Second Law

1K-1 Prove the rule (1) in Notes K for differentiating the dot product of two plane vectors: do the calculation using an \mathbf{i}, \mathbf{j} -coordinate system.

(Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{s}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$.)

1K-2 Let $\mathbf{s}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{s}}{dt} = \mathbf{0} \quad \Rightarrow \quad \mathbf{s}(t) = \mathbf{K}, \quad \text{where } \mathbf{K} \text{ is a constant vector.}$$

1K-3 In Notes K, by reversing the steps (5) – (8), prove the statement in the last paragraph. You will need the statement in exercise 1K-2.