M. Matrices and Linear Algebra

1. Matrix algebra.

In section D we calculated the determinants of square arrays of numbers. Such arrays are important in mathematics and its applications; they are called *matrices*. In general, they need not be square, only rectangular.

A rectangular array of numbers having m rows and n columns is called an $m \times n$ **matrix**. The number in the i-th row and j-th column (where $1 \le i \le m$, $1 \le j \le n$) is called the **ij-entry**, and denoted a_{ij}; the matrix itself is denoted by A, or sometimes by (a_{ij}).

Two matrices of the same size are *equal* if corresponding entries are equal.

Two special kinds of matrices are the **row-vectors**: the $1 \times n$ matrices $(a_1, a_2, \ldots, a_n)$; and the **column vectors**: the $m \times 1$ matrices consisting of a column of m numbers.

From now on, row-vectors or column-vectors will be indicated by boldface small letters; when writing them by hand, put an arrow over the symbol.

Matrix operations

There are four basic operations which produce new matrices from old.

1. Scalar multiplication: Multiply each entry by c : $cA = (ca_{ij})$

2. Matrix addition: Add the corresponding entries: $A + B = (a_{ij} + b_{ij})$; the two matrices must have the same number of rows and the same number of columns.

3. Transposition: The *transpose* of the $m \times n$ matrix A is the $n \times m$ matrix obtained by making the rows of A the columns of the new matrix. Common notations for the transpose are A^T and A'; using the first we can write its definition as $A^T = (a_{ji})$.

If the matrix A is square, you can think of A^T as the matrix obtained by flipping A over around its main diagonal.

Example 1.1 Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \\ -1 & 0 \end{pmatrix}$. Find $A+B$, A^T, $2A-3B$.

Solution. $A + B = \begin{pmatrix} 3 & 2 \\ -2 & 4 \\ -2 & 2 \end{pmatrix}$; $A^T = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \end{pmatrix}$;

$$2A + (-3B) = \begin{pmatrix} 4 & -6 \\ 0 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -15 \\ 6 & -9 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -21 \\ 6 & -7 \\ 1 & 4 \end{pmatrix}.$$

4. Matrix multiplication This is the most important operation. Schematically, we have

$$\begin{array}{ccccc} A & \cdot & B & = & C \\ m \times n & & n \times p & & m \times p \\ & & c_{ij} & = & \displaystyle\sum_{k=1}^{n} a_{ik} b_{kj} \end{array}$$

The essential points are:

1. For the multiplication to be defined, A must have as many *columns* as B has *rows*;

2. The ij-th entry of the product matrix C is the dot product of the i-th row of A with the j-th column of B.

Example 1.2 $\quad (2 \;\; 1 \;\; -1)\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = (-2+4-2) = (0)\,;$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(4 \;\; 5) = \begin{pmatrix} 4 & 5 \\ 8 & 10 \\ -4 & -5 \end{pmatrix}; \qquad \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \\ 0 & 2 & 2 \end{pmatrix}$$

The two most important types of multiplication, for multivariable calculus and differential equations, are:

1. AB, where A and B are two *square* matrices of the same size — these can always be multiplied;
2. $A\mathbf{b}$, where A is a square $n \times n$ matrix, and $\mathbf{b}$ is a column n-vector.

Laws and properties of matrix multiplication

M-1. $A(B+C) = AB + AC, \qquad (A+B)C = AC + BC$ — *distributive laws*

M-2. $(AB)C = A(BC); \qquad (cA)B = c(AB).$ — *associative laws*

In both cases, the matrices must have compatible dimensions.

M-3. Let $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; then $AI = A$ and $IA = A$ for any 3×3 matrix.

I is called the **identity** matrix of order 3. There is an analogously defined square identity matrix I_n of any order n, obeying the same multiplication laws.

M-4. In general, for two square $n \times n$ matrices A and B, $AB \neq BA$: *matrix multiplication is not commutative.* (There are a few important exceptions, but they are very special — for example, the equality $AI = IA$ where I is the identity matrix.)

M-5. For two square $n \times n$ matrices A and B, we have the *determinant law*:

$$|AB| = |A||B|, \qquad \text{also written} \qquad \det(AB) = \det(A)\det(B)$$

For 2×2 matrices, this can be verified by direct calculation, but this naive method is unsuitable for larger matrices; it's better to use some theory. We will simply assume it in these notes; we will also assume the other results above (of which only the associative law **M-2** offers any difficulty in the proof).

M-6. A useful fact is this: matrix multiplication can be used to pick out a row or column of a given matrix: you multiply by a simple row or column vector to do this. Two examples should give the idea:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \qquad \text{the second column}$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \qquad \text{the first row}$$

Exercises: Section 1F

2. Solving square systems of linear equations; inverse matrices.

Linear algebra is essentially about solving systems of linear equations, an important application of mathematics to real-world problems in engineering, business, and science, especially the social sciences. Here we will just stick to the most important case, where the system is *square*, i.e., there are as many variables as there are equations. In low dimensions such systems look as follows (we give a 2×2 system and a 3×3 system):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{7}$$

In these systems, the a_{ij} and b_i are given, and we want to solve for the x_i.

As a simple mathematical example, consider the linear change of coordinates given by the equations

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ x_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ x_3 &= a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{aligned}$$

If we know the y-coordinates of a point, then these equations tell us its x-coordinates immediately. But if instead we are given the x-coordinates, to find the y-coordinates we must solve a system of equations like (7) above, with the y_i as the unknowns.

Using matrix multiplication, we can abbreviate the system on the right in (7) by

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{8}$$

where A is the square matrix of coefficients (a_{ij}). (The 2×2 system and the $n \times n$ system would be written analogously; all of them are abbreviated by the same equation $A\mathbf{x} = \mathbf{b}$, notice.)

You have had experience with solving small systems like (7) by *elimination*: multiplying the equations by constants and subtracting them from each other, the purpose being to eliminate all the variables but one. When elimination is done systematically, it is an efficient method. Here however we want to talk about another method more compatible with hand-held calculators and MatLab, and which leads more rapidly to certain key ideas and results in linear algebra.

Inverse matrices.

Referring to the system (8), suppose we can find a square matrix M, the same size as A, such that

$$MA = I \qquad \text{(the identity matrix)}. \tag{9}$$

We can then solve (8) by matrix multiplication, using the successive steps,

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ M(A\mathbf{x}) &= M\mathbf{b} \\ \mathbf{x} &= M\mathbf{b}; \end{aligned} \tag{10}$$

where the step $M(A\mathbf{x}) = \mathbf{x}$ is justified by

$$\begin{aligned} M(A\mathbf{x}) &= (MA)\mathbf{x}, && \text{by M-2;} \\ &= I\mathbf{x}, && \text{by (9);} \\ &= \mathbf{x}, && \text{by M-3 .} \end{aligned}$$

Moreover, the solution is unique, since (10) gives an explicit formula for it.

The same procedure solves the problem of determining the inverse to the linear change of coordinates $\mathbf{x} = A\mathbf{y}$, as the next example illustrates.

Example 2.1 Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $M = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$. Verify that M satisfies (9) above, and use it to solve the first system below for x_i and the second for the y_i in terms of the x_i:

$$\begin{aligned} x_1 + 2x_2 &= -1 \\ 2x_1 + 3x_2 &= 4 \end{aligned} \qquad\qquad \begin{aligned} x_1 &= y_1 + 2y_2 \\ x_2 &= 2y_1 + 3y_2 \end{aligned}$$

Solution. We have $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, by matrix multiplication. To solve the first system, we have by (10), $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -6 \end{pmatrix}$, so the solution is $x_1 = 11, x_2 = -6$. By reasoning similar to that used above in going from $A\mathbf{x} = \mathbf{b}$ to $\mathbf{x} = M\mathbf{b}$, the solution to $\mathbf{x} = A\mathbf{y}$ is $\mathbf{y} = M\mathbf{x}$, so that we get

$$\begin{aligned} y_1 &= -3x_1 + 2x_2 \\ y_2 &= 2x_1 - x_2 \end{aligned}$$

as the expression for the y_i in terms of the x_i.

Our problem now is: how do we get the matrix M? In practice, you mostly press a key on the calculator, or type a Matlab command. But we need to be able to work abstractly with the matrix — i.e., with symbols, not just numbers, and for this some theoretical ideas are important. The first is that M doesn't always exist.

$$M \text{ exists} \quad \Leftrightarrow \quad |A| \neq 0.$$

The implication $\Rightarrow$ follows immediately from the law **M-5**, since

$$MA = I \quad \Rightarrow \quad |M||A| = |I| = 1 \quad \Rightarrow \quad |A| \neq 0.$$

The implication in the other direction requires more; for the low-dimensional cases, we will produce a formula for M. Let's go to the formal definition first, and give M its proper name, A^{-1}:

Definition. Let A be an $n \times n$ matrix, with $|A| \neq 0$. Then the **inverse** of A is an $n \times n$ matrix, written A^{-1}, such that

$$A^{-1}A = I_n, \qquad AA^{-1} = I_n \tag{11}$$

(It is actually enough to verify either equation; the other follows automatically — see the exercises.)

Using the above notation, our previous reasoning (9) - (10) shows that

$$|A| \neq 0 \quad \Rightarrow \quad \text{the unique solution of } A\mathbf{x} = \mathbf{b} \text{ is } \mathbf{x} = A^{-1}\mathbf{b}; \tag{12}$$

$$|A| \neq 0 \quad \Rightarrow \quad \text{the solution of } \mathbf{x} = A\mathbf{y} \text{ for the } y_i \text{ is } \mathbf{y} = A^{-1}\mathbf{x}. \tag{12}$$

Calculating the inverse of a 3×3 matrix

Let A be the matrix. The formulas for its **inverse** A^{-1} and for an auxiliary matrix $\operatorname{adj} A$ called the **adjoint** of A (or in some books the **adjugate** of A) are

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T . \tag{13}$$

In the formula, A_{ij} is the cofactor of the element a_{ij} in the matrix, i.e., its minor with its sign changed by the checkerboard rule (see section 1 on determinants).

Formula (13) shows that the steps in calculating the inverse matrix are:

1. Calculate the matrix of minors.
2. Change the signs of the entries according to the checkerboard rule.
3. Transpose the resulting matrix; this gives $\operatorname{adj} A$.
4. Divide every entry by $|A|$.

(If inconvenient, for example if it would produce a matrix having fractions for every entry, you can just leave the $1/|A|$ factor outside, as in the formula. Note that step 4 can only be taken if $|A| \neq 0$, so if you haven't checked this before, you'll be reminded of it now.)

The notation A_{ij} for a cofactor makes it look like a matrix, rather than a signed determinant; this isn't good, but we can live with it.

Example 2.2 Find the inverse to $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

Solution. We calculate that $|A| = 2$. Then the steps are (T means transpose):

$$\underset{\text{matrix } A}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \rightarrow \underset{\text{cofactor matrix}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}} \underset{\text{T}}{\rightarrow} \underset{\text{adj } A}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}} \rightarrow \underset{\text{inverse of } A}{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}$$

To get practice in matrix multiplication, check that $A \cdot A^{-1} = I$, or to avoid the fractions, check that $A \cdot \text{adj}\ (A) = 2I$.

The same procedure works for calculating the inverse of a 2×2 matrix A. We do it for a general matrix, since it will save you time in differential equations if you can learn the resulting formula.

$$\underset{\text{matrix } A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \rightarrow \underset{\text{cofactors}}{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}} \underset{\text{T}}{\rightarrow} \underset{\text{adj } A}{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \rightarrow \underset{\text{inverse of } A}{\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Example 2.3 Find the inverses to: a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Solution. a) Use the formula: $|A| = 2$, so $A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

b) Follow the previous scheme:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & -3 & 7 \\ 2 & 0 & -1 \\ 4 & 3 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} \rightarrow \frac{1}{3}\begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} = A^{-1}.$$

Both solutions should be checked by multiplying the answer by the respective A.

Proof of formula (13) for the inverse matrix.

We want to show $A \cdot A^{-1} = I$, or equivalently, $A \cdot \text{adj}\ A = |A|I$; when this last is written out using (13) (remembering to transpose the matrix on the right there), it becomes

$$(14) \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}.$$

To prove (14), it will be enough to look at two typical entries in the matrix on the right — say the first two in the top row. According to the rule for multiplying the two matrices on the left, what we have to show is that

$$(15) \qquad a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|;$$

$$(16) \qquad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

These two equations are both evaluating determinants by Laplace expansions: the first equation (15) evaluates the determinant on the left below by the cofactors of the first row; the second equation (16) evaluates the determinant on the right below by the cofactors of the second row (notice that the cofactors of the second row don't care what's actually in the second row, since to calculate them you only need to know the other two rows).

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The two equations (15) and (16) now follow, since the determinant on the left is just $|A|$, while the determinant on the right is 0, since two of its rows are the same. □

The procedure we have given for calculating an inverse works for $n \times n$ matrices, but gets to be too cumbersome if $n > 3$, and other methods are used. The calculation of A^{-1} for reasonable-sized A is a standard package in computer algebra programs and MatLab. Unfortunately, social scientists often want the inverses of very large matrices, and for this special techniques have had to be devised, which produce approximate but acceptable results.

Exercises: Section 1G

3. Cramer's rule *(some 18.02 classes omit this)*

The general square system and its solution may be written

$$A\mathbf{x} = \mathbf{b}, \qquad |A| \neq 0 \qquad \Rightarrow \qquad \mathbf{x} = A^{-1}\mathbf{b}\ . \tag{17}$$

When this solution is written out and simplified, it becomes a rule for solving the system $A\mathbf{x} = \mathbf{b}$ known as *Cramer's rule.* We illustrate with the 2×2 case first; the system is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad |A| \neq 0\ .$$

The solution is, according to (17),

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ a_{11}b_2 - a_{21}b_1 \end{pmatrix}.$$

If we write out the answer using determinants, it becomes **Cramer's rule**:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|}; \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|}\ . \tag{18}$$

The formulas in the 3×3 case are similar, and may be expressed this way:

Cramer's rule. *If $|A| \neq 0$, the solution of $A\mathbf{x} = \mathbf{b}$ is given by*

$$x_i = \frac{|A_i|}{|A|}, \qquad i = 1, 2, 3, \tag{19}$$

where $|A_i|$ is the determinant obtained by replacing the i-th column of $|A|$ by the column vector $\mathbf{b}$.

Cramer's rule is particularly useful if one only wants one of the x_i, as in the next example.

Example 3.1. Solve for x, using Cramer's rule (19):

$$\begin{aligned} 2x - 3y + z &= 1 \\ -x + y - z &= 2 \\ 4x + 3y - 2z &= -1 \,. \end{aligned}$$

Solution. We rewrite the system on the left below, then use Cramer's rule (19):

$$\begin{pmatrix} 2 & -3 & 1 \\ -1 & 1 & -1 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}; \qquad x = \frac{\begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ -1 & 1 & -1 \\ 4 & 3 & -2 \end{vmatrix}} = \frac{-7}{13} \,.$$

Proof of (19). Since the solution to the system is $\mathbf{x} = A^{-1}\mathbf{b}$, when we write it out explicitly, it becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} .$$

We show that this gives formula (19) for x_1; the arguments for the other x_i go similarly. From the definition of matrix multiplication, we get from the above

$$\begin{aligned} x_1 &= \frac{1}{|A|}(A_{11}b_1 + A_{21}b_2 + A_{31}b_3), \\ x_1 &= \frac{1}{|A|} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \end{aligned}$$

according to the Laplace expansion of the determinant by its first column. But this last equation is exactly Cramer's rule for finding x_1. □

Cramer's rule is also valid for $n \times n$ systems; it is not normally used for systems larger than 3×3 however. You would use A^{-1}, or systematic elimination of variables. Nonetheless, the formula (19) is important as a theoretical tool in proofs and derivations.

Exercises: Section 1H

4. Theorems about homogeneous and inhomogeneous systems.

On the basis of our work so far, we can formulate a few general results about square systems of linear equations. They are the theorems most frequently referred to in the applications.

Definition. The linear system $A\mathbf{x} = \mathbf{b}$ is called **homogeneous** if $\mathbf{b} = \mathbf{0}$; otherwise, it is called **inhomogeneous**.

Theorem 1. *Let A be an $n \times n$ matrix.*

(20) $$|A| \neq 0 \quad \Rightarrow \quad A\mathbf{x} = \mathbf{b} \text{ has the unique solution, } \mathbf{x} = A^{-1}\mathbf{b} .$$

(21) $$|A| \neq 0 \quad \Rightarrow \quad A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution, } \mathbf{x} = \mathbf{0}.$$

Notice that (21) is the special case of (20) where $\mathbf{b} = \mathbf{0}$. Often it is stated and used in the contrapositive form:

(21′) $$A\mathbf{x} = \mathbf{0} \text{ has a non-zero solution} \quad \Rightarrow \quad |A| = 0.$$

(The contrapositive of a statement $P \Rightarrow Q$ is not-$Q \Rightarrow$ not-P; the two statements say the same thing.)

Theorem 2. *Let A be an $n \times n$ matrix.*

(22) $$|A| = 0 \quad \Rightarrow \quad A\mathbf{x} = \mathbf{0} \text{ has non-trivial (i.e., non-zero) solutions.}$$

(23) $$|A| = 0 \quad \Rightarrow \quad A\mathbf{x} = \mathbf{b} \text{ usually has no solutions, but has solutions for some } \mathbf{b}.$$

In (23), we call the system **consistent** if it has solutions, **inconsistent** otherwise.

This probably seems like a maze of similar-sounding and confusing theorems. Let's get another perspective on these ideas by seeing how they apply separately to homogeneous and inhomogeneous systems.

Homogeneous systems: $A\mathbf{x} = \mathbf{0}$ *has non-trivial solutions* $\Leftrightarrow |A| = 0$.

Inhomogeneous systems: $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$, *if* $|A| \neq 0$. *If* $|A| = 0$, *then* $A\mathbf{x} = \mathbf{b}$ *usually has no solutions, but does have solutions for some* $\mathbf{b}$.

The statements (20) and (21) are proved, since we have a formula for the solution, and it is easy to see by multiplying $A\mathbf{x} = \mathbf{b}$ by A^{-1} that if $\mathbf{x}$ is a solution, it must be of the form $\mathbf{x} = A^{-1}\mathbf{b}$.

We prove (22) just for the 3×3 case, by interpreting it geometrically. We will give a partial argument for (23), based on both algebra and geometry.

Proof of (22).

We represent the three rows of A by the row vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and we let $\mathbf{x} = (x, y, z)$; think of all these as origin vectors, i.e., place their tails at the origin. Then, considering the homogeneous system first,

(24) $$A\mathbf{x} = \mathbf{0} \quad \text{is the same as the system} \quad \mathbf{a} \cdot \mathbf{x} = 0, \quad \mathbf{b} \cdot \mathbf{x} = 0, \quad \mathbf{c} \cdot \mathbf{x} = 0.$$

In other words, we are looking for a row vector $\mathbf{x}$ which is orthogonal to three given vectors, namely the three rows of A. By Section 1, we have

$$|A| = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \ = \ \text{volume of parallelepiped spanned by } \mathbf{a}, \mathbf{b}, \mathbf{c}.$$

It follows that if $|A| = 0$, the parallelepiped has zero volume, and therefore the origin vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in a plane. Any non-zero vector $\mathbf{x}$ which is orthogonal to this plane will then be orthogonal to $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and therefore will be a solution to the system (24). This proves (22): if $|A| = 0$, then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Partial proof of (23). We write the system as $A\mathbf{x} = \mathbf{d}$, where $\mathbf{d}$ is the column vector $\mathbf{d} = (d_1, d_2, d_3)^T$.

Writing this out as we did in (24), it becomes the system

$$\mathbf{a}\cdot\mathbf{x} = d_1, \quad \mathbf{b}\cdot\mathbf{x} = d_2, \quad \mathbf{c}\cdot\mathbf{x} = d_3 \ . \tag{25}$$

If $|A| = 0$, the three origin vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in a plane, which means we can write one of them, say $\mathbf{c}$, as a linear combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} = r\mathbf{a} + s\mathbf{b}, \qquad r.s \text{ real numbers.} \tag{26}$$

Then if $\mathbf{x}$ is any vector, it follows that

$$\mathbf{c}\cdot\mathbf{x} = r(\mathbf{a}\cdot\mathbf{x}) + s(\mathbf{b}\cdot\mathbf{x}) \ . \tag{27}$$

Now if $\mathbf{x}$ is also a solution to (25), we see from (25) and (27) that

$$d_3 = rd_1 + sd_2; \tag{28}$$

this shows that unless the components of $\mathbf{d}$ satisfy the relation (28), there cannot be a solution to (25); thus in general there are no solutions.

If however, $\mathbf{d}$ does satisfy the relation (28), then the last equation in (25) is a consequence of the first two and can be discarded, and we get a system of two equations in three unknowns, which will in general have a non-zero solution, unless they represent two planes which are parallel.

Singular matrices; computational difficulties.

Because so much depends on whether $|A|$ is zero or not, this property is given a name. We say the square matrix A is **singular** if $|A| = 0$, and **nonsingular** or **invertible** if $|A| \neq 0$.

> Indeed, we know that A^{-1} exists if and only if $|A| \neq 0$, which explains the term "invertible"; the use of "singular" will be familiar to Sherlock Holmes fans: it is the 19th century version of "peculiar" or the late 20th century word "special".

Even if A is nonsingular, the solution of $A\mathbf{x} = \mathbf{b}$ is likely to run into trouble if $|A| \approx 0$, or as one says, A is *almost-singular*. Namely, in the formulas given in Cramer's rule (19), the $|A|$ occurs in the denominator, so that unless there is some sort of compensation for this in the numerator, the solutions are likely to be very sensitive to small changes in the coefficients of A, i.e., to the coefficients of the equations. Systems (of any kind) whose solutions behave this way are said to be **ill-conditioned**; it is difficult to solve such systems numerically and special methods must be used.

To see the difficulty geometrically, think of a 2×2 system $A\mathbf{x} = \mathbf{b}$ as representing a pair of lines; the solution is the point in which they intersect. If $|A| \approx 0$, but its entries are not small, then its two rows must be vectors which are almost parallel (since they span a parallelogram of small area). The two lines are therefore almost parallel; their intersection point exists, but its position is highly sensitive to the exact positions of the two lines, i.e., to the values of the coefficients of the system of equations.

Exercises: Section 1H