18.02 - Practice Final A - Spring 2006

Problem 1. Let $P = (0,1,0)$, $Q = (2,1,3)$, $R = (1,-1,2)$. Compute $\overrightarrow{PQ} \times \overrightarrow{PR}$ and find the equation of the plane through P, Q, and R, in the form $ax + by + cz = d$.

Problem 2. Find the point of intersection of the line through $P_1 = (-1,2,-1)$ and $P_2 = (1,4,0)$ with the plane $3x - 2y + z = 1$.
Is P_2 on the same side of the plane as the origin $(0,0,0)$ or not?

Problem 3. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & c \\ 3 & c & 2 \end{bmatrix}$.

a) Find all values of c for which A is not invertible.

b) Let $c = 1$, and find the two entries marked $*$ in $A^{-1} = \begin{bmatrix} . & . & . \\ . & . & * \\ . & . & * \end{bmatrix}$.

Problem 4. Consider the plane curve given by $x(t) = e^t \cos t$, $y(t) = e^t \sin t$.

a) Find the velocity vector, and show that the speed is equal to $\sqrt{2}\, e^t$.

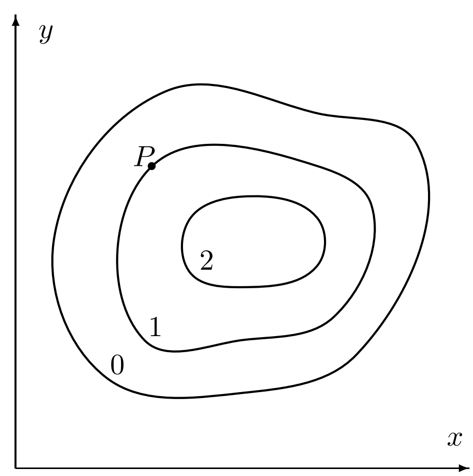
b) Find the angle between the velocity vector and the position vector, and show that it is the same for every t.

Problem 5. Let $f(x,y) = x^3 + xy^2 - 2y$.

a) Find the gradient of f at $(1,2)$ and use an approximation formula to estimate the value of $f(1.1, 1.9)$.

b) Use the chain rule to find the rate of change of f, df/dt, along the parametric curve $x(t) = t^3$, $y(t) = 2t^2$, at the time $t = 1$.

Problem 6. In the contour plot below: mark a point where $f = 1$, $f_x < 0$ and $f_y = 0$, and draw the direction of the gradient vector at the point P.



Problem 7. Let $f(x,y) = x^3 - xy + \frac{1}{2}y^2$.

a) Find all the critical points of f.

b) Determine the type of the critical point at the origin.

c) What are the maximum and the minimum of f in the region $x \geq 0$? (Justify your answer.)

Problem 8. a) Find the equation of the tangent plane to the surface $x^3 + yz = 1$ at $(-1, 2, 1)$.

b) Assume that x, y, z are constrained by the relation $x^3 + yz = 1$, and let f be a function of x, y, z whose gradient at $(-1, 2, 1)$ is $\langle a, b, c \rangle$. Find the value of $\left(\dfrac{\partial f}{\partial y}\right)_z$ at $(-1, 2, 1)$. Express your answer in terms of a, b, c.

Problem 9. Evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\, dy\, dx$ by changing the order of integration.

Problem 10. Evaluate the work done by the vector field $\mathbf{F} = -y^3\, i + x^3\, j$ around the circle of radius a centered at the origin, oriented counterclockwise in two ways: directly, or by using Green's theorem.

Problem 11. Find the flux of $x\hat{\imath}$ out of each side of the square of sidelength 2, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Explain why the total flux out of any square of sidelength 2 is the same regardless of its center and orientation.

Problem 12. Let $\mathbf{F} = (x^2 - xy)\hat{\imath} + 2y\hat{\jmath}$, and let C be the ellipse $(2x - y)^2 + (5x + y)^2 = 3$, oriented counterclockwise.
Use the normal form of Green's theorem to express the flux of $\mathbf{F}$ through C as a double integral.
(Give the integrand and region of integration, but do **not** provide limits for an iterated integral.)
Use a change of variables to evaluate the double integral you found.

Problem 13. Express the volume of the cylinder $0 \leq z \leq a$, $x^2 + y^2 \leq 1$ first as a triple integral in cylindrical coordinates and then as the sum of two triple integrals in spherical coordinates.

Problem 14. Let $\mathbf{F} = z^2\hat{\imath} + (z \sin y)\hat{\jmath} + (2z + axz + b\cos y)\hat{\boldsymbol{k}}$.

a) Find values of a and b such that $\mathbf{F}$ is conservative.

b) For these values of a and b, find a potential function for $\mathbf{F}$ using a systematic method.

c) Still using the same values of a and b you found in part (a), calculate $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ where C is the portion of the curve $x = t^3$, $y = 1 - t^2$, $z = t$ for $-1 \leq t \leq 1$.

Problem 15. Calculate the flux of $\mathbf{F} = x\hat{\imath} + y\hat{\jmath} + (1 - 2z)\hat{\boldsymbol{k}}$ out of the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$ in two ways: directly, or using the divergence theorem.

Problem 16. Let $\mathbf{F} = (-6y^2 + 6y)\hat{\imath} + (x^2 - 3z^2)\hat{\jmath} - x^2\hat{\boldsymbol{k}}$.
Calculate $\operatorname{curl} \mathbf{F}$ and use Stokes' theorem to show that the work done by $\mathbf{F}$ along any simple closed curve contained in the plane $x + 2y + z = 1$ is equal to zero.