Welcome back to recitation. In this video, I'd like us to do the following problem. I want us to show that the function, if I integrate x to the n, e to the minus x from 1 to infinity, that that actually converges for any value of n. And I want us to show this without using integration by parts n times. So I'd like you to figure out a way to show that this integral converges. And again, what we mean by converges, is for a fixed value here, say, some b, if I let- - the limit as b goes to infinity, that that sequence of values converges to a finite number. That's what we mean, again, when we say an integral converges. Ultimately, we want to show this is finite. So show this is finite, without using integration by parts n times. So I'll give you a little while to work on it, and then I'll be back, and I will show you how I did it.

OK. Welcome back. Well, I want to show you how we can show that this integral actually convergences for any n. And I don't want to have to use integration by parts and kill off powers of x in order to do that.

So again, the integral is from 1 to infinity of x to the n, e to the minus x dx. And if you recall, Professor Jerison was showing, in the lecture video, that if you can show even from, not 1 to infinity, but from very far out to infinity, that that integral converges, from 1 to whatever the far-out value is, this integral is finite. OK? Doesn't blow up. It's going to be potentially a very big number, but it is going to be a finite number. There are no places where we run into trouble with this function. It's a continuous function from 1 to infinity. So we can go very far out and say, OK, from very far out to infinity, the integral converges. And then that's going to be enough. So we did see that kind of technique earlier. But I just want to remind you, that's what we're going to do.

Now, you might have thought about this problem and said, well, I know that x to the n is much smaller than e to the x for values of x very large. So you might have said-- I think you used this notation also in the lecture. x to the n is much smaller than e to the x for large x.

So what if we tried to do a comparison with those two functions? We're going to see, that's not quite enough. But let's say x is very large, and let's look at a comparison. If I say, the integral from, say, some very large R to infinity of x to the n e to the minus x dx, it's certainly going to be much smaller than the integral from big R to infinity of e to the x times e to the minus x dx.

And you think, well, you know, that's a pretty good first step. I'm doing all right. But what happens here? What's e to the x times e to the minus x? It's e to the x plus negative x, so it's e to the 0, so it's 1. So this is integrating the constant 1. Well, the constant 1 from R to infinity, think of that. It's the line y equals 1 from R to infinity. It's an arbitrarily long rectangle. That's got a lot of area. It's got infinite area.

So this integral diverges. That doesn't mean this one diverges, right? Because this one is smaller than that one.

So this one here could still converge, even though this integral diverged. Again, let me remind you. Why does this integral diverge? Because this is actually equal to 1. If I integrate 1 from R to infinity, I get something infinite.

So you might have started with that, but that's not quite good enough. Right? What is going to be good enough, is if I pick any constant in front of this n-- I can put any constant in front of this n I want, bigger than 1. And that's going to help us out.

So I'm going to pick the constant 2, because it's going to be easy, and it's a nice fixed number. If I-- so this is how you do it correctly, or one strategy to do it correctly. If I take x to the 2n, instead of just x to the n, for any n, there's some R big enough so that x to the 2n is much less than e to the x for all x bigger than or equal to some R.

So if I go far enough out in x-values, x to the 2n is much smaller than e to the x. So let's say that that value is capital R, and then we're much smaller. This is not very formal, but it's getting closer to a formal kind of thing. Then that means x to the n is much smaller than e to the x over 2. Right? And that's going to be the key.

Let's anticipate why that is. The problem with the substitution of e to the x was that e to the x times e to the minus x gave you 1. But if I use e to the x over 2, I'm going to end up with some function, e to the minus something, and that's going to be good, because that's going to converge.

So if you tried e to the x first, and you saw you didn't get a good function, you didn't have to stop there. You could say, well, I was close. The problem is, I cancelled off all the e to the minus power, which is what I want to keep around. If I want to keep some of that around, then I have to have a little less power of e to the x there. I have to have-- I can't just have e to the x. I should have something like square root of e to the x. So e to the x over 2. That's going to help us out. So this is, this is kind of, as you're working on this type of problem, this is some of the thought process you want to go through.

So what do we see here? We have x to the n is much less than e to the x over 2 for x bigger than or equal to r. So now let's do a comparison with our new comparative function. So I'm going to come over, and this will be our last line to finish this off. So now we're integrating from R to infinity x to the n, e to the minus x dx. And we know that's going to be much less than integral from R to infinity, e to the x over 2, e to the minus x dx.

And now let's figure out what this is. e to the minus x plus x over 2 is e to the minus x over 2. And the good news is, this is, we know this converges. I'll check, and I'll show you, remind you that it converges. But we know this converges. And the reason is, because e to the minus x is a function that decays so fast as it goes to 0. That's really why you get the convergence.

Actually, you could even compare this to x to the minus 2 right away. And you could get something like, you know this decays faster than x to the minus 2, and we know x to the minus 2 converges. So you could even compare it to that. You could do a second comparison in here. But I'll actually calculate this, just to remind us how we do this.

So e to the minus x over 2. If I want to find an antiderivative, I'm going to guess and check. I know I'm going to have an e to the minus x over 2 again, and then I need to be able to kill off a negative 1/2. So I should put a negative 2 in front. Let's double check. This is basically a substitution problem. An easy substitution.

So e to the minus x over 2. Its derivative is negative 1/2 x, and then itself again. The negative 1/2 times negative 2 gives me a 1. So again, I'm just, it's an easy substitution problem, but I always want to check. So I evaluate that from R to infinity. Well, the point is, e to the minus infinity-- this is 0. As x goes to infinity, this quantity goes to 0. So I get 0 minus a negative 2, so plus 2, e to the minus R over 2. For some fixed, big R. Well, that's finite. That's a finite number. So we come back here. This integral converges. That integral was this integral. And this integral, then, is bigger than this one. So this one converges.

Now, this had a lot of pieces to it, so I'm going to remind us sort of what was happening. So let's go back to the original function, and I'll just take us back through one more time.

OK. So the original problem was, show that this integral from 1 to infinity of x to the n e to the minus x dx converges. And I reminded you that you knew from lecture that if I could show that it converged for some very large number down here to infinity, that was sufficient, because this function is continuous from 1 to infinity. I don't have to worry about places where I might get infinite area in a finite interval. I'm always going to have finite area in a finite interval. So if I start at some big R to infinity and that converges, then I'm good. Because from 1 to R, that'll be finite.

So then the point is, you want to compare. And I mentioned a comparison that doesn't quite work, but is a good first test. Because you know x to the n is much less than e to the x for a sufficiently large x. You might think to compare it to that, but the problem again was when we do that substitution, we actually get an integral that diverges. But a divergent integral bigger than something doesn't mean this one diverges. If the divergence was-- the inequality was the other way, then you could show this one diverged. But we actually show convergence this way. Or we're trying to show convergence in this direction, so we need to say, if this one converges, then that one converges. This diverging doesn't tell us anything about this.

So then we say, OK. This one didn't work, but it almost worked. So what if I figure out a way to compare x to the n to a slightly smaller thing than e to the x? And that slightly smaller thing is e to the x over 2. It's not really slightly smaller. But the smaller function is e to the x over 2. OK?

And this is a way to think about how that works. Is that for any power of x to the 2n, I can still get it smaller than e to the x for some sufficiently large number. OK?

And you don't even have to think about these R's as being the same. I can change them, I can make this bigger. This doesn't compare to this problem, also. So don't be confused by those two R's.

OK. So then we found something we wanted to compare x to the n with. Then we come back over here, and we actually see that we get a good comparison, because we're able to see that the integral on the right-hand side converges. This integral is bigger than this integral. So this integral converges, so we know this integral converges. And then, the integral from 1 to R of this is finite, so the integral from 1 to infinity of this converges. And that's sort of the strategy for doing these types of problems.

OK. That's where I'll stop.