

**JOEL LEWIS:** Hi, welcome back to recitation. In lecture, Professor Jerison was teaching about L'Hospital's Rule, and one thing that he mentioned several times was that when you apply L'Hospital's Rule, it's really important that the second limit exists in order for L'Hospital's Rule to be true, in order for the first limit to equal the second limit. So I have a-- but he didn't give you any examples of what can go wrong.

So I have here a limit for you to work on. So what I'd like you to do is try L'Hospital's Rule, see what happens, and then try to solve it a different way not using L'Hospital's Rule. So why don't you pause the video, spend a couple minutes working on that, come back, and we can work on it together.

Welcome back. Hopefully you've have some fun working on this limit. Let's go through it and see what happens when we try to apply L'Hospital's Rule to it. So the first thing to notice is that as  $x$  goes to infinity,  $x$  plus cosine  $x$ -- well, cosine  $x$  is small, and you know, between minus 1 and 1--  $x$  is going to infinity. So the top is going to infinity and the bottom is going to infinity. So this is an infinity over infinity indeterminate form. So it is a context in which we can try to apply L'Hospital's Rule, so that's fine. So it's an indeterminate ratio, we can look at applying L'Hospital's Rule.

So let's try to do that. So L'Hospital's Rule says that the limit as  $x$  goes to infinity of our expression  $x$  plus cosine  $x$  divided by  $x$  is equal-- and I'm going to put a little question mark here because what L'Hospital's Rule says is that it's equal provided-- it's equal to what I'm going to write over here, provided that the second limit exists.

So what goes over here? Well it's the limit as  $x$  goes to the same place, as  $x$  goes to infinity, of the ratio of the derivative, so the derivative of the top divided by the derivative of the bottom. So in this case that's  $1$  minus sine  $x$  is the derivative of the top and  $1$  is the derivative of the bottom.

OK, now let's look at this limit. What happens? Well, OK the over  $1$  part is irrelevant. The  $1$  minus part, OK, who cares? The only thing that's changing in this limit is the sine  $x$ . So as  $x$  goes to infinity what this does is it oscillates, just like sine  $x$  does. I mean, it's offset and flipped upside down because of this  $1$  minus.

So, in particular, you know, sometimes it's near  $1$ , sometimes it's near negative  $1$ , sometimes

it's near 0, and it oscillates back and forth in between. So in particular, because it's oscillating, it's not approaching any value. So this limit doesn't exist. It's not equal to any real number. It's not equal to infinity. It's not equal to minus infinity. It does not exist.

OK, so the statement of L'Hospital's theorem says that this equality is really wrong. OK? What it means is that L'Hospital's Rule tells you nothing. You don't learn anything here. So if we want to compute this first limit, we can't use L'Hospital's Rule. We have to come up with some better way to do it, OK. So that's what's going wrong.

Now let's see about doing this even though we don't have this tool. So if we try and solve this limit without L'Hospital's Rule, so we want to look at the limit as  $x$  goes to infinity of  $x$  plus cosine  $x$  divided by  $x$ . Well, think about what's important in this limit. As  $x$  is getting big, well  $x$  is getting big and  $x$  is getting big and what's cosine  $x$  doing? Well cosine  $x$  is behaving like a constant but wigglier, right, we could say.

So cosine  $x$  is always between minus 1 and 1, but, you know, it's oscillating, but it's in some bounded interval there, whereas  $x$  and  $x$  are both going to infinity at exactly the same speed, so this suggests a manipulation that sort of allows us to quantify that explicitly.

And so one thing that we can do is we can divide top and bottom of this limit by  $x$ . So if we do that, or equivalently we can just divide this  $x$  into the fraction above, split it into two fractions. So we have-- I'm just going to re-write it one more time-- the limit as  $x$  goes to infinity of  $x$  plus cosine  $x$  divided by  $x$  is equal to the limit as  $x$  goes to infinity of  $1$  plus cosine  $x$  over  $x$ .

All right, so what's happening here? Well, all I've done is I've divided that  $x$  in-- and now you see that this  $1$ , that was from the  $x$  over  $x$ , so that's a constant that's going to  $1$ . And as  $x$  goes to infinity, cosine  $x$ -- again we can think of it like a constant but wigglier--  $x$  is going to infinity, so cosine  $x$  over  $x$  is going to  $0$  as  $x$  goes to infinity. So I guess I wouldn't call this plugging in, but it's, you know, sort of just like plugging in. So this part is going to  $0$ , so we're left with just  $1$ .

OK, so we can evaluate this limit fairly easily. It just took a couple steps. But we can't evaluate it with L'Hospital's Rule. So this is why you have that extra thing in the statement of the theorem that Professor Jerison kept saying, and that I'll say every time I use it, which is L'Hospital's Rule says that the limit of the ratio is equal to the limit of the ratio of the derivative provided the second limit exists.

OK, so here's an example where the second limit didn't exist and so we can't say that the two

are actually equal to each other, and we need some other tools to evaluate the limit in question. So I'll end there.