Exploring a Parametric Curve

a) Describe the curve traced out by the parametrization:

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t, \end{aligned}$$

where $0 \le t \le 4\pi$.

b) Set up and simplify, but do not integrate, an expression for the arc length $\int_0^{4\pi} \frac{ds}{dt}\, dt$ of this curve.

Solution

a) Describe the curve traced out by the parametrization

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t, \end{aligned}$$

where $0 \le t \le 4\pi$.

We know that the equations:

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}$$

describe a circle of radius a, where a is a constant.

In the problem we've been given, the multiple a is variable:

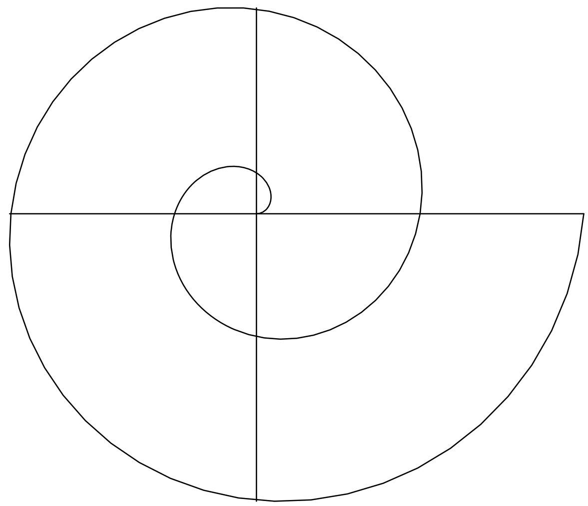
$$\begin{aligned} x &= a(t)\cos t \\ y &= a(t)\sin t, \end{aligned}$$

where $a(t) = t$. The curve still moves counter-clockwise around the origin as t increases; however, its distance from the origin also increases as t increases.

We could compute the coordinates of several points on the curve to get a better idea of its behavior. The table below gives a few of these coordinates.

t	(x, y)
0	$(0, 0)$
$\frac{\pi}{4}$	$\left(\frac{\pi\sqrt{2}}{8}, \frac{\pi\sqrt{2}}{8}\right)$
$\frac{\pi}{2}$	$\left(0, \frac{\pi}{2}\right)$
π	$(-\pi, 0)$
$\frac{3\pi}{2}$	$\left(0, -\frac{3\pi}{2}\right)$

We see that the curve starts at $(0, 0)$ and proceeds counter-clockwise, moving away from the origin. It wraps twice around the origin as t increases from 0 to 4π, tracing out a figure known as an Archimedean spiral.



b) Set up and simplify, but do not integrate, an expression for the arc length $\int_0^{4\pi} \frac{ds}{dt}\,dt$ of this curve.

We know that $ds = \sqrt{dx^2 + dy^2}$, so:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

We compute that:

$$\begin{aligned}
\frac{dx}{dt} &= -t\sin t + \cos t \\
\frac{dy}{dt} &= t\cos t + \sin t \\
\left(\frac{dx}{dt}\right)^2 &= t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t \\
\left(\frac{dy}{dt}\right)^2 &= t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t \\
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2(\sin^2 t + \cos^2 t) - 2t\sin t\cos t + 2t\sin t\cos t + (\cos^2 t + \sin^2 t) \\
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 + 1.
\end{aligned}$$

Thus we have:

$$\text{Arc length} = \int_0^{4\pi} \frac{ds}{dt}\,dt$$

$$= \int_0^{4\pi} \sqrt{t^2+1}\,dt.$$

If we wished we could complete this calculation using the inverse substitution $t = \tan\theta$ and then integrating $\sec^3\theta$. We could then check our work by comparing our final result to the circumference of an appropriate circle.