Simpson's Rule

This approach often yields much more accurate results than the trapezoidal rule does. Again we divide the area under the curve into n equal parts, but for this rule n must be an even number because we're estimating the areas of regions of width $2\Delta x$.

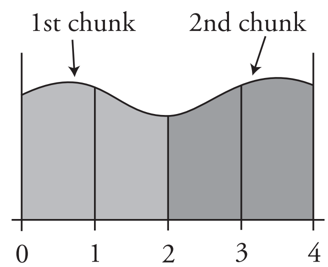

Figure 1: Simpson's rule for n intervals (n must be even!)

When computing Riemann sums, we approximated the height of the graph by a constant function. Using the trapezoidal rule we used a linear approximation to the graph. With Simpson's rule we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. When Δx is small this approximates the curve very closely, and we get a fantastic numerical approximation of the definite integral.

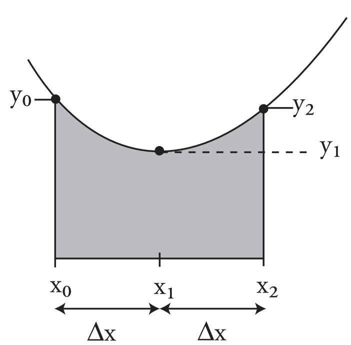

Figure 2: Using a parabolic approximation of the curve.

The derivation of the formula for Simpson's Rule is left as an exercise, but the area of this region is essentially the base times some average height of the graph:

$$\text{Area} = (\text{base})(\text{average height}) = (2\Delta x)\left(\frac{y_o + 4y_1 + y_2}{6}\right).$$

This emphasizes the middle more than the sides, which is consistent with the equations for parabolic approximation.

Simpson's rule gives you the following estimate for the area under the curve:

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)\left[(y_o + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (\cdots y_n)\right].$$

We can combine terms here by exploiting the following pattern in the coefficients:

1	4	1				
		1	4	1		
				1	4	1
1	4	2	4	1	4	1

To get the final form of Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$