Solids of Revolution

In theory we could take any three dimensional object and estimate its volume by slicing it into slabs and adding the volumes of the slabs. In practice we'll concentrate exclusively on *solids of revolution*. These are formed by taking an area — for example the arc over the x-axis shown in Figure 1 — and revolving it about an axis to see what volume it sweeps out. If you rotate that arc and its interior about the x-axis you get a shape like an American football or a rugby ball. (See Figure 2.)

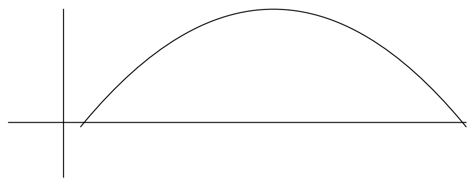

Figure 1: An arc over the x-axis.

Method of Disks

How would we slice up this ball to find its volume? We'll start with the two dimensional picture of an arc over the x-axis. Two dimensional figures are much easier to draw and understand than three dimensional ones; when possible you should avoid drawing three dimensional figures. In this picture we draw a thin rectangle whose base lies on the x-axis and whose height is the height of the arc. The width of this rectangle is dx.

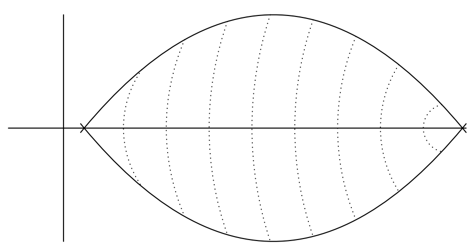

Figure 2: The solid formed by revolving an arc about the x-axis.

Next we need to visualize how this rectangle is related to the three dimensional volume of revolution. When we rotate the arc about the x-axis, the rectangle rotates also as if it were hinged. As it rotates it sweeps out a disk or coin shape. This corresponds to one "slice" of our solid ball, like a slice of bread. The method that we're describing for figuring out the volume of the ball is called the *method of disks* because we're slicing the ball into disks.

We add the volumes of these disks to find the volume of the ball. If the height of the arc over the x-axis is y, then the area $A(x)$ of a face of the disk or slice is πy^2 because the rectangle swept out a circular shape as it spun about the x-axis. The thickness of the disk is dx, so the volume of the disk is

$$dV = (\pi y^2)\, dx.$$

This will be the integrand in the formula for volume associated with the method of disks.

$$V = \int (\pi y^2)\, dx$$

Notice that we have both a y and a dx in our formula, yet we don't have a formula describing y in terms of x. That formula depends on the equation $y = f(x)$ of the arc over the x-axis, and will change depending on the situation.

In addition, we haven't specified any limits of integration yet; again those will depend on the situation.