PROFESSOR: Welcome back to recitation. Today we're going to work on a problem involving differential equations. I'm going to read it to you, give you a little hint, give you some time to work on it, and then I'll be back and work it out for you.

So, the problem is to find a function y equals f of x that has the following two properties: d squared y dx squared is equal to 6x. And just to remind you what this means, this is really, this is the second derivative of y with respect to x. So, the second derivative with respect to x should be 6x. And the second condition's kind of long, but it says the graph of the function in the xy-plane passes through the point (1, 1) with a horizontal tangent there. So let me give you one hint. And that hint is that there are some initial conditions buried in here. That's why we have this condition. So I'm going to give you a little bit of time to work on it and I'll be back and I'll work it out with you.

Welcome back. Hopefully you were able to start at least solving the problem initially, give yourself a little direction. So let's see how you did. OK, so the first thing I would like to do is try and figure out maybe what the first derivative of the function y equals f of x is. I have its second derivative, so in order to find the first derivative I want to find, ultimately, a function that when I take its derivative I get 6x. Right?

So, we can think about this. Maybe the easiest thing for us to do would actually be to consider another function whose derivative is 6x. And we'll know that's dy dx. So I'm going to say this: dy-- sorry-- d squared y dx squared, I'm going to say is the first derivative of another function, we'll say dw/dx. And the reason I'm going to do that is so we're not too nervous about how we solve this problem. So let's just assume that. So I'm introducing another function w, which is the first derivative of y with respect to x, ultimately. Because its derivative is the second derivative of y with respect to x.

So let's see how to solve the differential equation dw/dx equals 6x. Now you may be able to do that right away. You may see what this is right away. If you're a little nervous, we can do a separation of variables. So right away, maybe some of you can see that this will be 2x-- sorry-- 3 x squared. But let's just double-check. So, separation of variables, on this side we get the integral of dw. On this side we get the integral of 6x dx. Here we get a w. And here, again we get, we should really do 6 x squared over 2 plus a constant. So that's 3 x squared plus a constant.

I'm going to write c_1 here because we're went to need a little bit of information. We're going to need another constant later. So we actually now know the derivative of y with respect to x. What we found here, I'll just write that in, is dy/dx. Again, let me remind you why. We had-- we were saying the second derivative of y with respect to x we're going to call dw/dx. So when I took an integral of dw/dx, I got w. So that means I've taken one antiderivative here and so now I have dy/dx. So this is dy/dx.

I'm going to draw a line and now we want to find y. And again we can use separation of variables. Or this time I'm just going to do the problem without separating variables, because I actually know what an antiderivative is of this, what an antiderivative is of this, and then I need to add one more constant.

So I can say that y is definitely equal to x cubed plus c_1*x plus c_2. And let me just again, let's see why that is. Right? This antiderivative of this is x cubed. Antiderivative of this is-- it's a constant so its antiderivative is c_1 times x. And then I have to add on another constant because I have a whole family of possible solutions.

So here I have y. And now I need to figure out how to use number two. Maybe before you even go on, you want to check, does this really satisfy number one? So if we wanted to check that, we just take two derivatives of this expression on the right-hand side. Two derivatives of this is 0. Two derivative of this is 0. And two derivatives of this is 6x. So I do indeed get what I want. So now we definitely can go on to number 2. OK?

So, what do we have? Number two, it says we have some initial conditions here. The graph in the xy-plane passes through the point (1, 1) and it has a horizontal tangent there. Now what does that actually mean? Let's think about that. That actually means that two says f of 1 equals 1. Right? y is equal to f of x. So we can write this also as f of x. So two, the first part says that f of 1 is 1.

And what does this second condition? Let's check this second condition. It says it has a horizontal tangent there. Horizontal tangent means that its derivative at that x-value is 0. So let me write down that in a nice form. The derivative at that x-value is equal to 0.

This is a little different, maybe, from what we've seen previously. In the lecture you saw examples, at least-- certainly, where you had one initial condition. Here we need two initial conditions. And you see why, is because we actually have two constants. Where did that come from? It's because we started with a second derivative instead of just the first derivative. So that's kind of, that's why you see more initial conditions than maybe you've seen previously.

So, let's plug these in and let's see what we get. If f of 1 equals 1, then let's evaluate that. f of 1 is equal to 1 cubed plus 1 times c_1 plus 1 times-- or just 1, or c_2 there, sorry. There's no x there, so just c_2. And that all has to equal 1. And then let's look at what the derivative is. The derivative is still over here. So f prime at 1 is equal to 3 times 1 squared, so 3 times 1, plus c_1. And the condition says that equals 0, equals 0.

So we can read off. The nice thing is this is system of equations but one of them is very easy to solve initially, then we can substitute. So what does this say? Well 3 plus c_1 equals 0. So c_1 is equal to minus 3, negative 3. If I plug in negative 3 for c_1, in this expression, in this equation up here, I get negative 3 plus 1 plus c_2 has to equal 1. If I subtract the 1's from both sides I get negative 3 plus c_2 has to equal 0. So c_2 has to actually equal 3.

So the final, final answer is evaluating, or plugging in the c1 and the c2 in for the constants there. The final, final answer is x cubed minus 3x plus 3. So we started with a differential equation and two sets of initial conditions. And we came up with one solution that satisfies that. Now you can look at this, you could graph this on a calculator or computer and look and see if it satisfies that it actually passes through the point (1, 1) and has a horizontal tangent there. But we know, based on our work, that that actually should happen. So I think that's where I'll stop.