Exponential Growth and Inhibited Growth

a) The differential equation $\dfrac{dy}{dx} = ry$ describes a situation in which a population size y increases at a rate proportional to its size. Use separation of variables to find a solution to this equation.

b) The differential equation $\dfrac{dy}{dx} = ry(s - y)$ $(s > 0)$ describes change in a population which tends toward a fixed size s. For example, this might describe a population in which food or space is limited. Use separation of variables and the fact that $\displaystyle\int \dfrac{dy}{y(s - y)} = \dfrac{1}{s} \ln\left|\dfrac{y}{s - y}\right| + c$ to find a solution to this equation.

Solution

a) The differential equation $\dfrac{dy}{dx} = ry$ describes a situation in which a population size y increases at a rate proportional to its size. Use separation of variables to find a solution to this equation.

The solution to this problem is similar to the example presented in lecture. We separate the variables, integrate both sides, and then simplify the logarithmic equation that results.

$$
\begin{aligned}
\frac{dy}{dx} &= ry \\
\frac{dy}{y} &= r\,dx \\
\int \frac{dy}{y} &= \int r\,dx \\
\ln|y| &= rx + c \\
|y| &= e^{rx+c} \\
|y| &= e^c e^{rx} \\
y &= ae^{rx}
\end{aligned}
$$

Note that when $x = 0$, $y = a$. If the variable x represents time, the equation $y = ae^{rx}$ gives the size of a population at time x if the initial population was a and the rate of change of the population is r times the current size of the population.

b) The differential equation $\dfrac{dy}{dx} = ry(s - y)$ $(s > 0)$ describes change in a population which tends toward a fixed size s. For example, this might describe a population in which food or space is limited. Use separation of variables

and the fact that $\int \frac{dy}{y(s-y)} = \frac{1}{s}\ln\left|\frac{y}{s-y}\right| + c$ to find a solution to this equation.

Before we start solving the problem, let's think about what the equation $\frac{dy}{dx} = ry(s-y)$ says about the rate of change in population. Notice that $y(s-y)$ is positive for $0 < y < s$ and negative otherwise. If the size y of the population is greater than s, the rate of change $\frac{dy}{dx}$ is negative and the population shrinks until $y \leq s$.

This differential equation can also be solved by separation of variables, but the antidifferentiation involved is more difficult.

$$
\begin{aligned}
\frac{dy}{dx} &= ry(s-y) \\
\frac{dy}{y(s-y)} &= r\,dx \qquad \text{(separate variables)} \\
\int \frac{dy}{y(s-y)} &= \int r\,dx \qquad \text{(integrate both sides)} \\
\frac{1}{s}\ln\left(\frac{y}{s-y}\right) &= rx + c \qquad \text{(use the hint provided)} \\
\ln\left|\frac{y}{s-y}\right| &= s(rx+c) \qquad \text{(simplify)} \\
\left|\frac{y}{s-y}\right| &= e^{srx+c'} \\
\frac{y}{s-y} &= ae^{srx} \\
y &= ae^{srx}(s-y) \\
y + yae^{srx} &= sae^{srx} \\
y &= \frac{sae^{srx}}{1 + ae^{srx}}
\end{aligned}
$$

Note that as the value of x approaches infinity, the value of y approaches s. Our original analysis of the differential equation suggested that the value of y would increase when it was below s and decrease if was above s, so it is not surprising to find that the value of y tends toward the value of s.

18.01SC Single Variable Calculus
Fall 2010