

PROFESSOR: Hi. Welcome back to recitation. We've been talking about antidifferentiation or integration by substitution and also by a method that Professor Jerison called "advanced guessing". So I have a few problems up on the board behind me. Three antiderivatives for you to compute.

So the first one is e^{2x} times cosine of the quantity $1 - e^{2x}$ dx . The second one is $4x$ times the quantity $5x^2 - 1$ raised to the $1/3$ power. And the third one is \tan of x dx . So why that you spend a few minutes, try to compute those antiderivatives yourself, come back and we can see how you did.

All right, so welcome back. We have these three antiderivatives, so let's take them in order. So this first one that I wrote is the antiderivative of e^{2x} times cosine of the quantity $1 - e^{2x}$ dx . So this problem seems to me like a good candidate for a substitution. So we have this clear sort of nested function thing going on. We have cosine of $1 - e^{2x}$. And then out front we have something that looks a lot like the derivative of this $1 - e^{2x}$.

So I'm going to try this with substitution then. And I think there are, you know, a few choices of substitution but a natural one is to sort of find the most complicated inside piece. So in this case, that's this whole thing, $1 - e^{2x}$. So I'm going to take, for my substitution, I'm going to take u equals $1 - e^{2x}$. And so that means du is equal to $-2e^{2x} dx$. OK, so that's my substitution. And when I put my substitution into this integral, what do I get? Well, so I have cosine of $1 - e^{2x}$ just becomes cosine of u . Now $e^{2x} dx$, that's very, very close to this du here. So what's different is that here I have a minus 2. So actually $e^{2x} dx$ is du divided by minus 2. So this is cosine u times du divided by minus 2.

Now another way to get to this point is you could solve this equation for dx and substitute it in and also solve this equation for e^{2x} and substitute it in and things should work out more or less the same if you try that out. So actually you won't even need to make that second substitution. You'll just get some nice cancellation there. It's even simpler than what I just said.

OK, so we do this antiderivative, we've made this substitution. So now we have just the antiderivative of a cosine function. All right, well that's not that bad. Because we know that the derivative of sine is cosine. So the antiderivative of cosine is sine. So this is equal to-- so that minus 2, that 1 over minus 2 is just going to stick around. So it's 1 over minus 2 sine of u plus

a constant of course, plus an arbitrary constant. And OK, and so, but my original function was in terms of x , so I want to bring everything back in terms of x . And so I need to substitute back in, get rid of u and replace it with x . So here that's a-- I'll just go back to what my substitution was and I replace all my u 's with it. So this is minus $\frac{1}{2}$ sine the quantity $1 - e$ to the $2x$ plus a constant. All right, so this is the antiderivative of this first expression here.

OK, so now, how about the second one? So the second one, we could also do it with a substitution. This is also sort of a prime suspect for advanced guessing. So we see here that we have some, this polynomial raised to some power. So this is $5x^2 - 1$ to the $\frac{1}{3}$. So how can we get from a derivative, something like $5x^2 - 1$ quantity to the $\frac{1}{3}$? Well if you started off with $5x^2 - 1$ to the $\frac{4}{3}$ and took a derivative, you would have this $5x^2 - 1$ to the $\frac{1}{3}$ coming out. And you would also have some stuff coming out in front by the chain rule.

Well what kind of stuff? Well you know, it would be some derivative of this quadratic polynomial, which would be some linear polynomial. And indeed, that kind of matches what we have out front. So a good guess for advanced guessing, is that we can look at-- so d over dx of $5x^2 - 1$, quantity, to the $\frac{4}{3}$. So OK, so this derivative we can compute by the chain rule. So this is $\frac{4}{3}$ times $5x^2 - 1$ to the $\frac{1}{3}$ times-- OK, so now I need to do the chain rule, I need to take the derivative of the inside. Well that's-- OK, so the minus 1 gets killed by the derivative, so $5x^2$, that becomes $10x$. So I can rewrite this as $\frac{40x}{3}$ times $5x^2 - 1$ to the $\frac{1}{3}$.

So this looks very much like the thing that we were interested in. Right? We were-- where'd it go? Oh, here it is. And the thing we were trying to antidifferentiate. So the difference is just this constant out front is a little bit different. So here I have 4, whereas when I took this derivative I had $\frac{40}{3}$. So I need to correct for that. And the correction is just to say, instead of starting with this $5x^2 - 1$ to the $\frac{4}{3}$, I need to start with some multiple of it to make the constant work out right in the end.

So in this case I was off by a multiple of $\frac{10}{3}$, so I need to correct by multiplying through by $\frac{3}{10}$. So we get that the antiderivative that we want. The antiderivative of $4x$ times the quantity $5x^2 - 1$ to the $\frac{1}{3}$ dx is equal to, well it's equal to $\frac{3}{10}$ of this. $5x^2 - 1$ to the $\frac{4}{3}$. OK, so that's our second antiderivative, which we got by advanced guessing.

Now let's look at the third one. So the third one is $\tan x$. Now I sort of promised you by asking

this question in this section on substitution that there's, you know, some substitution you can make. But it's not sort of obvious just from looking at $\tan x$ what should be substituted where. At least it isn't obvious to me. But one thing that can help often when you don't immediately see a substitution, is to try rewriting things in equivalent ways. So sometimes you can do some algebra or some other manipulation.

In this case there's a very simple sort of rewriting that you can do. Which is that tangent of x can be expressed in terms of sine and cosine of x . So we can rewrite the antiderivative of $\tan x$ dx as $\int \frac{\sin x}{\cos x} dx$. OK, so now what do we see? So, I see in the denominator a cosine of x . And then up top I have a sine x dx . So sine x dx , that's really close to the differential of cosine of x .

So I'm going to try this substitution. And I'm going to try the substitution u equals cosine x . So if I make the substitution I get du is equal to minus sine x dx . Which is, OK, so now if I plug these values in with this substitution, this integral becomes the integral, well it's minus 1 or minus du over u . That's a nice simple antiderivative to have. So we've seen this before. So this is just a logarithm. So the minus sine comes out front. So this is minus \ln of the absolute value of u plus a constant. And now we had this, that u was cosine of x . So this is minus \ln of the absolute value of cosine of x plus a constant.

Now this should look a little bit familiar. Because in one of Christine's recitations earlier, she had you compute the derivative of \ln of cosine x . And in that case you saw that that derivative was equal to minus tangent of x . Just like it should be. So, all right, so there we go. That was three examples of antidifferentiation by substitution and advanced guessing. So I'll leave you with that.