

PROFESSOR: Welcome back to recitation.

Today what I want to do is something maybe a little bit more theoretical, but the goal is to show that something that you are going to be repeatedly doing when you use quadratic approximations is, in fact, true. So I'm going to explain the situation, give a quick example, and then show you what we're setting out to do.

So the situation is as follows: we're going to-- any time you see a Q of f , that's going to represent the quadratic approximation to f at x equals 0. So what I've done is say, Q of f I'm going to define to be the thing on the right, which is exactly the formula you were given in class for the quadratic approximation of a function f at x equals 0. So f is approximately the thing on the right near 0. Our goal is to show that if I want to take the quadratic approximation of a product of two functions, that I can do it in a different way. I can do it in the way written on the right hand side, which actually looks more complicated in this notation, but is, in fact, easier in reality. So let me explain what's happening and then I'll give you an example.

If I wanted to take the quadratic approximation of a product of two functions, what I want to show is that instead, I could take the quadratic approximation of each individual function, multiply those together, and then take the quadratic approximation of what I get as a result.

So let me give you an easy example. For instance, let's let f of x equal e to the x and let's let g of x equal $\sin x$. Then what is Q of f ? Q of f is the quadratic approximation to e to the x at x equals 0. And that's going to be $1 + x + \frac{x^2}{2}$ -- you already knew this. And g , the quadratic approximation of $\sin x$, is just x .

So if I wanted to find the quadratic approximation to e to the $x \sin x$, what I could do-- what this is claiming I can do-- is instead I can take the quadratic approximation of this function times this function. So instead I can take the quadratic approximation of-- f was the e to the x -- $1 + x + \frac{x^2}{2}$, times x . That's not a 4, sorry. That's a parentheses. $1 + x + \frac{x^2}{2}$, times x . And what is that? The quadratic approximation to that is the quadratic approximation to $x + \frac{x^3}{2}$. And at x equals 0, if I have a polynomial, the quadratic approximation to a polynomial at x equals 0 is just all the terms up to the quadratic term. So I drop off higher-order terms. So I just get $x + \frac{x^2}{2}$.

So that's the idea. The idea is I have a product of two functions, I know their individual

quadratic approximations, and so what I want to do is I want to find the quadratic approximation of this product by exploiting the fact that I already know their individual ones, and explain the fact that quadratic approximation of polynomials at $x = 0$ is very easy. So that's the example, that's the idea.

So now let's see if we can do it. OK. So we have a cheat sheet up here that I'm going to refer back to. I didn't want to use it again and I didn't want to have to derive this for you, but we have the product rule: $f'g$ prime is equal to what's on the right, and $f''g$ double prime is equal to what's on the right. So my goal here is to show, remember, that the quadratic approximation-- let me point over here again. The goal is to show the quadratic approximation of $f'g$ is equal to the quadratic approximation of quadratic approximation of f times the quadratic approximation of g .

So let's do, well let's do the right-hand side first because that's a little nicer. And then we'll show the right-hand side and then we'll show the left-hand side and we'll show they're equal.

So let me start here with the right-hand side. OK? So, let's look at what's the quadratic approximation of f and what's the quadratic approximation of g and then we'll take their final quadratic approximation. So Q of f , we have exactly what we need there. f of 0 plus f' prime of 0 times x plus f'' double prime of 0 over $2x^2$. Q of g is equal to g of 0 plus g' prime of 0 times x plus g'' double prime at 0 over $2x^2$.

So now what I'm going to do is multiply those two together. And I'm actually going to swing this way a tiny bit, if that's OK, to write Q of f times Q of g because it's going to be a little long. And I'm going to group them carefully so that I have all the higher-order terms at the end. OK? So I'm going to get f of 0 g of 0 by multiplying these two together. And then I'm going to get two terms involving an x . I'm going to get an f' prime times g and a g' prime times f .

Actually, if you'll allow me, we'll know that anywhere we see an f or a g or an f' prime or a g' prime, or an f'' double prime or g'' double prime, they're all evaluated at 0 . So I'm going to drop the 0 's from here on out. Anywhere you see those, I'm evaluating them at 0 . Otherwise this will be way too long.

So let me write this just as $f'g$. I can even write just a single one evaluated at 0 . It's the product evaluated at 0 . And then I have f' prime g -- so I'll just evaluate it at 0 at the end of the product-- plus $f g'$ prime evaluated at 0 . This whole thing is times x . I get an x here, I get an x here.

Now I need to figure out what terms give me an x squared. OK? So the terms that give me an x squared are $f(0) \times g''(0)/2$. That gives me an x squared. $f''(0) \times g(0)$ gives me an x squared. And $f'(0) \times g'(0)$ gives me an x squared. So let's write out those terms. I get $f(0) \times g''(0)/2$ from those two-- plus $f''(0) \times g(0)$ from those two-- plus $f'(0) \times g'(0)$, all times x squared. So there's an x times an x there, gives you an x squared. x squared there, x squared there.

Now I could keep going, and I will mention the higher-order terms, but I'm not going to write them all the way out because of what we're about to do. Let me show you where they come from. You get an x cubed term from here and an x cubed term from here. Maybe I'll write the x cubed, but I won't write the x to the fourth. So the x cubed terms are $f'(0) \times g''(0)/2$ plus $f''(0) \times g'(0)$ over 2. And those are my x cubed terms.

And then I got some x to the fourth terms. And where do the x to the fourth terms come from? They come from this product. Right? But I want to point out something. What I'm going to do, I'm going to work some magic on the board. This is a quadratic approximation of f times a quadratic approximation of g . Let me come over here and remind you that I want the quadratic approximation of that product. So what I'm going to do is go back and look at what I need from there, to be the quadratic approximation of that product.

So we come back over here. If I apply the quadratic approximation to this thing, which means then I'm applying it to this whole giant thing, what do I actually get? This is actually a polynomial. I have something-- I have a linear term, I have an x , I have an x squared, I have an x cubed, I have an x to the fourth.

So if I apply that quadratic approximation, let's see what stays. This term stays. This term stays. This term stays. I'm going to erase what disappears because I don't want us to get confused by that. So these two terms disappear. Why? Because again, this is a polynomial. I have linear-- or I have constant, I'm sorry, I think I called this maybe linear earlier-- constant, linear, quadratic term. And what I need is just those if I'm looking for a quadratic approximation. So constant, linear, quadratic term. I immediately drop the cubic and the quartic term when I'm looking at a quadratic approximation of a polynomial at 0. So if I want the right-hand side, I just need what's underlined in blue. So now I'm going to put a big box around that because that's going to be important. Whatever else happens, we don't lose that. So now we've done the right-hand side.

And now let's write out what is the left-hand side. And that's just going to be plugging it straight into the formula. And using our cheat sheet.

So Q of $f \cdot g$, let me write out the definition and then we'll use the cheat sheet. It's $f \cdot g$ at 0 -- again, this is f at 0 , g at 0 , that's what this notation means-- plus $f \cdot g$ quantity prime at 0 times x plus quantity $f \cdot g$ double prime at 0 over 2 times x squared.

And now what we're hoping, remember, is that what's in the box is what shows up over here. Because this is the long way to do the problem. This would be if I took either the $x \sin x$ and I took all the derivatives. And this, in fact, even though it looks more confusing in evaluating such a quadratic approximation, this way would be the easier way. We just want to show we don't lose anything by doing what would be the easier way.

So I get $f \cdot g$ at 0 . And that's good-- we can see we already have one of those, so that's nice. What do I get here? I get f prime g at 0 plus g prime f at 0 times x . That comes-- let me remind you, I'll walk over here-- comes from the cheat sheet. The first thing. $f \cdot g$ prime is f prime g plus g prime f . We know that one pretty well, but just to remind you. So that's where this term comes from. This looks promising because if we come back to our quadratic of quadratic of f times quadratic of g , it looks exactly like the second term here. So now we're hoping that the x squared term looks like this. The only thing that might make you nervous is this doesn't have an over 2 , but if you were paying attention to the cheat sheet you'll probably see where that's coming. And I'll point it out in one moment.

So $f \cdot g$ double prime, using the cheat sheet, is f double prime g plus g double prime f plus $2 g$ prime f prime. I should have put 0 's in there. Just to be consistent let me put these 0 's in there. $2 f$ prime g prime at 0 . And then I have to divide the whole thing by 2 because there's a divided by 2 there, and then times x squared.

So let me move out of the way for a moment. So this numerator came from the cheat sheet for the second derivative. And if you need, we can go back. Let me just remind you, here it is. You can work it out for yourself. You can just take the derivative of the first derivative. But that's where this comes from. So let me go back and we're almost done with the problem.

So what do we see? Well, we see that we get $f \cdot g$ at 0 . We get the second term we want, f prime g at 0 plus g prime f at 0 times x . And then the third term is, in fact, exactly what we want because we get f double prime g at 0 over 2 plus g double prime f at 0 over 2 plus-- the 2 's divide out-- and I get f prime g prime at 0 time x squared. And if we look at this last term

and we look at the squared term in the box we see, in fact, they are exactly the same.

So let me summarize because this was kind of a long video. So I'm going to go back to the beginning, give you the example, and tell you what we were really trying to do here. So let's come back over here. And let me remind you, the goal was to show that if I wanted to take a quadratic approximation of a product of two functions, if I already knew their individual quadratic approximations, you were told that you could take those two quadratic approximations, multiply them, and drop off the higher-order terms. The higher order than 2.

So we had an example. We knew these two quadratic approximations. And you've been told that quadratic approximation of their product is just the quadratic approximation of the product of their quadratic approximations. And so our goal today was to show that you don't drop any of the terms that you get if you do it by this method or by this method. And we've done that.

So I think I'll stop there.